MATH410 - 0401

Ash Dorsey
ash@ash.lgbt

Contents

Preliminaries

Set Operations

Let A and B be sets.

Intersection

$A \cap B = \{x \mid x \in A \land x \in B\}$

Union

$A \cup B = \{x \mid x \in A \lor x \in B\}$

Singleton

A set containing only one element

Universal Set

Denoted U, all sets are subsets of U.

Properties of union and intersection

1. $A \cap B \subset A$
2. $A \cap B \subset B$
3. $A \subset A \cup B$
4. $B \subset A \cup B$

Complement

If there is a universal set, A^C is the set of all elements in the universal set but not in A.

Minus

Denoted $A \setminus B$, it is all the elements in A that are not in the elements in B.

Furthermore, $A \setminus B = A \cap B^C$.

Subset

A set A is a subset of a set B if every element in A is also in B.

For example, if $A = \{1, 2, 3\}, B = \{a, b, c, 1, 2, 3\}$, $A \subset B$.

Empty Set

The empty set has nothing in it. It is denoted $\varnothing$.

Functions

A function from a set A to a set B associates an element of B with each element of A.

A is called the domain, and B is called the codomain.

$$f : A \to B$$
$$f(x) = 2x + 3$$

If $A = \{1, 2, 4\}$, $\{5, 7, 11\} \subset B$.

For each element $x \in A$, f associates x with someone element $y \in B$.

We write $f(x) = y$. We say f maps x to y.

Example

$$f(x) = \frac{1}{x-2}$$
$$f : \mathbb{R} \setminus \{2\} \to \mathbb{R}$$

Image

Let there be a function $f : A \to B$

$$f(A) = \{y \,|\, y = f(x) \land x \in A\}$$

The image of A is then $f(A)$

Injective/One-to-one

$f : A \to B$ is injective if:

$\forall x_1, x_2 \in A, f(x_1) = f(x_2) \to x_1 = x_2$

Therefore, if $f(x) = y$, then x is the only element of A that f maps to y.

Example

$f(x) = 3x + 1$ is one-to-one, $f(x) = x^2$, is not one-to-one

Surjective/Onto

$f : A \to B$ is a surjection if $f(A) = B$.

$\forall x \in B, \exists y \in A, x = f(y)$

Bijective

If f is injective and surjective, then f is bijective.

Set Cardinality

A set's cardinality is the number of elements in the set.

Two sets A and B have the same cardinality if there exists a bijection function $f : A \to B$.

Example

$\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{N}$ and $\mathbb{A}$ have the same cardinality, called countably infinite.

Existence of Inverses

An inverse of a function exists iff a function is bijective. The inverse of f is denoted f^{-1}.

Axioms of Real Numbers

$\mathbb{R}$ are built based on three axioms:

1. Field Axioms
    1. Commutativity of addition: $\forall a, b \in \mathbb{R}, a + b = b + a$

2. Associativity of addition: $\forall a, b, c \in \mathbb{R}, a + (b + c) = (a + b) + c$
3. Additive identity: there is a real number, denoted 0, such that $\forall a \in \mathbb{R}, 0 + a = a + 0 = a$
4. Additive inverse: for each $a \in \mathbb{R}, \exists b \in \mathbb{R}$ so that $a + b = 0$.
5. Commutativity of multiplication: $\forall a, b \in \mathbb{R}, a \times b = b \times a$
6. Associativity of multiplication: $\forall a, b, c \in \mathbb{R}, a \times (b \times c) = (a \times b) \times c$
7. Multiplicative identity: there is a real number, denoted 1, such that $\forall a \in \mathbb{R}, 1 \times a = a \times 1 = a$
8. Multiplicative inverse: $\forall a \in \mathbb{R}, a \neq 0 \to \exists b \in \mathbb{R}, ab = 1$
9. Distributive property: $\forall a, b, c \in \mathbb{R}, a(b + c) = ab + ac$.
10. Nontriviality, $0 \neq 1$.

2. Positivity axiom
   - How Reals are ordered
3. Completeness axiom
   - Reals have no gaps

Proposition 1

The element 0 is the only real number satisfying the additive identity property.

Proof

Suppose $\exists z \in \mathbb{R}$, so that $\forall a \in \mathbb{R}, z + a = a$.

Let b be the additive inverse of a.

$$\begin{aligned} & z \\ &= z + 0 \\ &= z + (a + b) \\ &= (z + a) + b \\ &= a + b \\ &= 0 \end{aligned}$$

Therefore, $z = 0$.

Proposition 2

The element 1 is the only real number satisfying the multiplicative identity property.

Proof

Suppose $\exists z \in \mathbb{R}$, so that $\forall a \in \mathbb{R}, z \times a = a$.

Assume $a \neq 0$, and then let b be the multiplicative inverse of a. Then,

$$\begin{aligned} & z \\ &= 1 \times z \\ &= a \times b \times z \\ &= z \times a \times b \\ &= a \times b \\ &= 1 \end{aligned}$$

Proposition 3

$\forall a \in \mathbb{R}, 0a = a0 = 0$

Proof

Let $a \in \mathbb{R}$, and let b be the additive inverse of a.

$$\begin{aligned} & 0 \\ & = a + b \\ & = 1a + b \\ & = (1 + 0)a + b \\ & = 1a + 0a + b \\ & = a + 0a + b \\ & = a + b + 0a \\ & = 0 + 0a \\ & = 0a \\ & = a0 \end{aligned}$$

Proposition 4

If $a, b \in \mathbb{R}$ such that $ab = 0$, $a = 0 \vee b = 0$

Proof

Let $a, b \in \mathbb{R}$, such that $ab = 0$.

If $a = 0$, $a = 0 \vee b = 0$ is true, because $a = 0$.

Otherwise, WLOG, suppose $a \neq 0$.

Then, there is a multiplicative inverse c of a such that $ca = 1$.

$$\begin{aligned} & 0 \\ & = c0 \\ & = c(ab) \\ & = (ca)b \\ & = 1b \\ & = b \end{aligned}$$

Therefore $b = 0$, and similarily $b \neq 0 \to a = 0$, and so $a = 0 \vee b = 0$ is always true.

Proposition 5

$\forall a \in \mathbb{R}$ there is a unique solution x to the equation $a + x = 0$.

Proof

Let $a \in \mathbb{R}$.

Let b be the additive inverse of a. Therefore, $x = b$ is a solution.

By contradiction, suppose $x = y$ also a solution to $a + x = 0$, $b \neq y$.

$$
\begin{aligned}
& b \\
& = b + 0 \\
& = b + (a + y) \\
& = (b + a) + y \\
& = (a + b) + y \\
& = 0 + y \\
& = y
\end{aligned}
$$

But by assumption $b \neq y$, so our assumption was wrong, and therefore there is only one solution x to $a + x = 0$.

Since each $a \in \mathbb{R}$ has a unique additive inverse, we denote it by $-a$ and define subtraction by $a - b = a + (-b)$

A similar proof show for any nonzero $a \in \mathbb{R}$, we have a unique multplicative inverse, denoted by a^{-1}, and define the quotient of a and b as

$$\frac{a}{b} = ab^{-1}$$

Exercises

Prove the following:

1. $-(-a) = a$
2. $\left(b^{-1}\right)^{-1} = b$
3. $(-b)^{-1} = -\left(b^{-1}\right)$
4. $(-a)b = -(ab)$
5. $\left(ab^{-1}\right)^{-1} = a^{-1}b$

Positivity Axioms of Real Numbers

There is a subset $\mathbb{P}$, called positive numbers, of real numbers with the following properties:

1. $a, b \in \mathbb{P} \to ab, a + b \in \mathbb{P}$
2. $\forall a \in \mathbb{R}$, exactly one of the following is true:
   1. $a \in \mathbb{P}$
   2. $-a \in \mathbb{P}$
   3. $a = 0$

These axioms let us define $>$ or $<$ operators.

Let $a, b \in \mathbb{R}$.

1. $a > b$ if $a - b \in \mathbb{P}$
2. $a < b$ if $b - a \in \mathbb{P}$
3. $a = b$ if $a - b = 0$

Then, $a \in \mathbb{P} \leftrightarrow a > 0$.

Proposition

If $a \in \mathbb{R} \setminus \{0\}$, then $a^2 > 0$.

Proposition

If $a \in \mathbb{P}$, then $a^{-1} \in \mathbb{P}$

Proposition

If $a > b$, $(c > 0 \to ac > bc) \wedge (c < 0 \to ac < bc)$.

Interval Notation

Let $a < b$, then we define

1. $(a, b) = \{x \in \mathbb{R} \,|\, a < x < b\}$
2. $[a, b] = \{x \in \mathbb{R} \,|\, a \leq x \leq b\}$
3. $(a, b] = \{x \in \mathbb{R} \,|\, a < x \leq b\}$
4. $[a, b) = \{x \in \mathbb{R} \,|\, a \leq x < b\}$

Inductive Set

A set S is inductive iff

1. $1 \in S$
2. $\forall x \in S, x + 1 \in S$

Natural Numbers: $\mathbb{N}$

Let I be the collection of all inductive sets.

Define:

$$\mathbb{N} = \bigcap_{S \in I} S$$

Proposition

The natural numbers $\mathbb{N}$ are inductive

Proof

First point:

$$\forall S \in I, 1 \in S \to 1 \in \bigcap_{S \in I} S$$

Second property:

Let $x \in \mathbb{N}$. Then $\forall S \in I, x \in S$ and $\forall S \in I, x + 1 \in \mathbb{S}$

Therefore, $\mathbb{N}$ is inductive.

Properties of $\mathbb{N}$

For $n, m \in \mathbb{N}$:

1. $m + n \in \mathbb{N}$
2. $m \times n \in \mathbb{N}$

Integers: $\mathbb{Z}$

$$\mathbb{Z} = \mathbb{N} \cup \{0\} \cup \{-x \,|\, x \in \mathbb{N}\}$$

Properties of $\mathbb{Z}$

For $n, m \in \mathbb{Z}$:

1. $m + n \in \mathbb{Z}$
2. $m - n \in \mathbb{Z}$
3. $m \times n \in \mathbb{Z}$

Rationals: $\mathbb{Q}$

$$\mathbb{Q} = \left\{ \frac{n}{m} \,\middle|\, n, m \in \mathbb{Z}, m \neq 0 \right\}$$

Proposition

$\mathbb{Q}$ satisfies the field and positivity axioms.

Proposition

$$\forall a, b \in \mathbb{Q}, a < b, \exists c \in \mathbb{Q}, a < c < b$$

Proof:

Let $c = \frac{a+b}{2}$. $c \in \mathbb{Q}$ by the field axioms.

$$c - a = \frac{a+b}{2} - a = \frac{b-a}{2} = \frac{1}{2}(b-a) > 0$$

Facts

1. Each rational number can be written as $\frac{m}{n}$ where either $2 \nmid m \vee 2 \nmid n.$.
2. An integer n is even iff n^2 is even.

Formational Example

$$f(x) = x^2 - 2$$

$f(1) < 0, f(2) > 0$ It would be good if $\exists c, 1 < c < 2, f(c) = 0$.

But this is impossible in the rationals.

Proposition

There is no rational number a so that $a^2 = 2$.

Proof

On the contrary, assume $a \in \mathbb{Q}$ where $a^2 = 2$.

Therefore, $a = \frac{m}{n}, m, n \in \mathbb{Z}$. By a fact we know that either m or n is odd.

$$a = \frac{m}{n}$$

$$2 = a^2 = \frac{m^2}{n^2}$$

$$2n^2 = m^2$$

Therefore, m^2 is even, which means that m is even.

Since m is even, $\exists k \in \mathbb{Z}, m = 2k$.

Thus,

$$
\begin{aligned}
2n^2 &= m^2 \\
2n^2 &= 4k^2 \\
n^2 &= 2k^2
\end{aligned}
$$

Therefore n^2 is even, which means that n is even.

Both n and m are even, which is a contradiction.

Therefore, there is no $a \in \mathbb{Q}$ with $a^2 = 2$.

Tools of Analysis

Bounded Above

A nonempty $S \subset \mathbb{R}$ is bounded above if

$$\exists c \in \mathbb{R}, \forall x \in S, x \leq c$$

c is then an upper bound.

Bounded Below

A nonempty $S \subset \mathbb{R}$ is bounded below if

$$\exists c \in \mathbb{R}, \forall x \in S, c \leq x$$

c is then a lower bound.

Completeness Axiom

Every $S \subset \mathbb{R}$ that is bounded above has a least upper bound c so that $\forall x \in S, x \leq c$ and if b is an upper bound of S, $c \leq b$.

Definition of Real Numbers

Any set that satisfies the
- Field axioms
- Positivity Axoims
- Completeness axiom

is equivalent to $\mathbb{R}$

Supremum

The supremum of a set is the least upper bound of that set.

Example

$\sup([2, 3)) = 3$.

Infimum

Every $S \subset \mathbb{R}$ bounded below has a greatest lower bound, $c = \inf S$

$\sqrt{2} \in \mathbb{R}$
IOW, $\exists a \in \mathbb{R}^{\geq 0}, a^2 = 2$

Proof

Let $S = \{x \in \mathbb{R} \,|\, x \geq 0, x^2 \leq 2\}$.

2 is an upper bound. Let $a = \sup(S)$

On the contrary, assume $a^2 > 2$. Let $r = \frac{a^2-2}{2a}$. $r > 0$.

$$(a-r)^2 = a^2 - 2ar + r^2 \overset{r>0}{>} a^2 - 2ar = a^2 - \cancel{2a}\frac{a^2-2}{\cancel{2a}} = 2$$

$$\forall x \in S, (a-r)^2 > 2, a - r > x$$
$$a - r < a, a \leq 2.$$

(incomplete)

Archimedean Property

1. For all $c \in \mathbb{R}, c > 0$, there exists $n \in \mathbb{N}$ with $n > c$
2. For all $\varepsilon \in \mathbb{R}, \varepsilon > 0$, there exists $n \in \mathbb{N}$ with $\frac{1}{n} < \varepsilon$

(these statements are equivalent.)

Proof

We will prove the first statement.

On the contrary, assume $\exists c \in \mathbb{R}, \forall n \in \mathbb{N}, n \leq c$.

Let $b = \sup \mathbb{N}$. This must exist because c is an upper bound of $\mathbb{N}$. Since b is the least upper bound, $b - \frac{1}{2}$ is not an upper bound. Therefore, $\exists n \in \mathbb{N}, n > b - \frac{1}{2}$.

Then, $n + 1 > \left(b - \frac{1}{2}\right) + 1 = b + \frac{1}{2} > b$. Therefore, $n + 1 > b$, but $n + 1 \in \mathbb{N}$. Therefore, c does not exist, and so the first statement is true.

Integers will not exist between integers

Let $n \in \mathbb{Z}$. There is no integer in $(n, n+1)$.

Proof

Consider the set $\{k \,|\, k \in \mathbb{N}, k \geq 1\}$. It is an inductive subset of $\mathbb{N}$ (by the positivity axioms), and therefore it is $\mathbb{N}$. Therefore, $\forall k \in \mathbb{N}, k \geq 1$.

Since all positive integers are in $\mathbb{N}$, the interval $(0, 1) \cap \mathbb{N} = \varnothing$.

Now suppose $k \in (n, n+1) \cap \mathbb{Z}$. Then, $n < k < n + 1$. Therefore, $0 < k - n < 1$. $k - n \in \mathbb{Z}$. Since $k - n > 0$ and $k \in \mathbb{Z}$, $k \in \mathbb{N}$. Therefore, $k - n \in (0, 1) \cap \mathbb{N}$, but $(0, 1) \cap \mathbb{N} = \varnothing$, so k does not exist, and there is a contradiction.

Sets of integers have maxima

If S is a nonempty set of integers that is bounded above, then S has a maximum.

Proof

Let $a = \sup S$. a is the least upper bound of S. Therefore, $a - 1$ is not an upper bound. Therefore, $\exists n \in S, a - 1 < n$. Therefore, $a < n + 1$. Then, $S \subset (-\infty, n+1)$. By the previous result, $(n, n+1)$ contains no elements.

$(-\infty, n+1) = (-\infty, n] \cup (n, n+1)$. Therefore, $S \subset (-\infty, n]$, and $n \in S$, so n is the maximum of S.

One integer exist in each interval of size 1

For any $c \in \mathbb{R}$, $\exists! n \in \mathbb{N}, n \in [c, c+1)$

Proof

Let $S = \{n \,|\, n \in \mathbb{Z}, n < c+1\}$.

If $c \geq 0$, then $0 \in S$.

If $c < 0$, by the Archimedean property, $\exists m \in \mathbb{N}, m > -c$. Thus, $-m < c < c+1$, therefore $-m \in S$.

Therefore $S \neq \varnothing$.

By the previous result, S has a maximum n.

By defintion of S, $n < c+1$.

If $n < c \to n+1 < c+1$ and therefore $n+1 \in S$. But this is impossible since n was the max of S and therefore $n \geq c$.

Therefore, $c \leq n < c+1$ and so $n \in [c, c+1)$.

Let $n, m \in \mathbb{Z} \cap [c, c+1)$.

WLOG, assume $m \leq n$:

$$0 \leq n - m < (c+1) - c = 1$$
$$0 \leq n - m < 1$$

So $n - m \in [0, 1) \cap \mathbb{Z}$ and therefore $n - m = 0 \to n = m$.

A rational exists between any two reals

For any $a, b \in \mathbb{R}$ with $a < b, \exists c \in \mathbb{Q}$ with $a < c < b$, and therefore $c \in (a, b)$.

Proof

Let $\frac{1}{n} < b - a$. Then: (This is incomplete)

$$nb - 1 \leq m \leq nb$$
$$a < b - \frac{1}{n} \leq \frac{m}{n} < b$$

Dense

A set $S \subset \mathbb{R}$ is dense iff $\forall a, b \in \mathbb{R}, a < b \to S \cap (a, b) \neq \varnothing$.

Examples

$\mathbb{Q}, \mathbb{R} \setminus \mathbb{Q}, \mathbb{A}, \mathbb{Q} \setminus \mathbb{Z}$

Absolute Value

$$|\cdot| : \mathbb{R} \to \{x \in \mathbb{R} \,|\, x \geq 0\}$$

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Properties

1. If $d \geq 0$, $|c| \leq d$ iff $d \leq c \leq d$.
2. $\forall x \in \mathbb{R}, -|x| \leq x \leq |x|$

Triangle Inequality

$$\forall a, b, \in \mathbb{R}, |a + b| \leq |a| + |b|$$

Proof

$$-|a| \leq a \leq |a|$$

$$-|b| \leq b \leq |b|$$

$$-|a| - |b| \leq a + b \leq |a| + |b|$$

$$-(|a| + |b|) \leq a + b \leq |a| + |b|$$

$$\leftrightarrow |a + b| \leq |a| + |b|$$

Some Sums

$$a^n - b^n = (a - b) \sum_{k=0}^{n-1} a^{n-1-k} b^k$$

$$\forall r \neq 0, \sum_{k=0}^{n} r^k = \frac{1 - r^{n+1}}{1 - r}$$

Sequences

Sequences

A sequence is some function $a : \mathbb{N} \to \mathbb{R}$. This is typically written as some a_n rather than $a(n)$.

The entire sequence is denoted like $\{a_n\}$.[1]

Examples

[1]Or perhaps (a_n) or $(a_n)_{n=1}^{\infty}$

$$\{n^2\} = \{1, 4, 9, 16, ...\}$$
$$\{1 + (-1)^n\} = \{0, 2, 0, 2, 0, 2, ...\}$$
$$\{a_n\} \text{ where } a_n \in \left(0, \frac{1}{n}\right)$$
$$a_1 = 1, a_{n+1} = 3a_n + 1, \{a_n\} = \{1, 4, 13, ...\}$$
$$\{a_n\} = \left\{1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, ...\right\}$$
$$\{a_n\} = \left\{\sum_{k=0}^{n} r^k\right\}$$
$$\{a_n\} = \left\{\sum_{k=1}^{n} \frac{1}{k}\right\}$$

Convergence of Sequence

A sequence $\{a_n\}$ converges to $a \in \mathbb{R}$ if $\forall \varepsilon > 0, \exists N, n \geq N \to |a_n - a| < \varepsilon$.

Therefore, we say:

$$\lim_{n \to \infty} a_n = a$$

Example: Converge

Prove $\{a_n\} = \left\{\frac{1}{n}\right\}$ converges to $a = 0$.

Fix $\varepsilon > 0$.

Let $N > \frac{1}{\varepsilon}$. Therefore, $n \geq N > \frac{1}{\varepsilon}$ and:

$$n > \frac{1}{\varepsilon}$$
$$\frac{1}{n} < \varepsilon$$
$$a_n < \varepsilon$$
$$|a_n| < \varepsilon$$
$$|a_n - a| < \varepsilon$$
$$\therefore \square$$

Example: Does not converge

Prove $\{a_n\} = \{1 + (-1)^n\}$ does not converge.

Assume $\{a_n\}$ converges to a. Let $\varepsilon = 1$, $a \in \mathbb{R}$, let $N > 0$. Let n_1 be the smallest even number larger than N and let n_2 be the smallest odd number larger than N.

$$|2 - a| = \left|a_{n_1} - a\right| < \varepsilon$$
$$|a| = |0 - a| = \left|a_{n_2} - a\right| < \varepsilon$$

$$\begin{aligned} 2 &= |2 - a + a| \\ &\leq |2 - a| + |a| \\ &< |2 - a| + \varepsilon \\ &< \varepsilon + \varepsilon \\ &< 2 \end{aligned}$$

But $2 < 2$ is false, so it does not converge.

Cannot converge to two different values

$$\left(\lim_{n\to\infty} a_n = a \wedge \lim_{n\to\infty} a_n = b\right) \to (a = b)$$

Proof

Fix $\varepsilon > 0$. Then $\exists N_1$ such that if $n \geq N$ then $|a_n - a| < \frac{\varepsilon}{2}$, and $\exists N_2$ such that $n \geq N_2$, then $|a_n - b| < \varepsilon$.

Consider $\varepsilon = b - a$. (WLOG, $b > a$).

Let $N > \max\{N_1, N_2\}$. Then if $n \geq N$:

$$\begin{aligned} \varepsilon &= |b - a| \\ &= |b - a_n + a_n - a| \\ &\leq |b - a_n| + |a_n - a| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon \end{aligned}$$

But $\varepsilon < \varepsilon$ is impossible, and so $b > a$ is false, and therefore $a \neq b$.

Create Limit to Zero

$$\{a_n\} \to a \leftrightarrow \{a_n - a\} \to 0$$

Proof

Prove $\to$ direction. Fix $\varepsilon > 0$. $\exists N > 0, n \geq N \to |a_n - a| < \varepsilon$.

Therefore, $|(a_n - a) - 0| < \varepsilon$.

Prove $\leftarrow$ direction. Fix $\varepsilon > 0$. $\exists N > 0, n \geq N \to |(a_n - a) - 0| < \varepsilon$.

Therefore, $|a_n - a| < \varepsilon$.

Therefore $\leftrightarrow$ is proven

All Convergent Sequences are Bounded

If $\{a_n\}$ converges, then $\exists M \geq 0, \forall n, |a_n| \leq M$

Proof

Let a be the limit of $\{a_n\}$.

Choose $\varepsilon = 1$.

$\exists N, n \geq N \to |a_n - a| < 1$.

Let $M = \max\{a+1, |a_1|, |a_2|, ..., |a_{N-1}|\}$

Clearly, $\forall n \in \mathbb{N} \cap [1, N-1], M \geq |a_n|$.

If $n \geq N$, then $M - a_n = M - a_n - a + a$.

Then

$$\begin{aligned} M - a_n &= M - a_n + a - a \\ &= (M - a) - (a_n - a) \\ &> M - a - 1 \\ &= M - (a + 1) \\ &\geq 0 \end{aligned}$$

Comparison Lemma

Suppose $\{a_n\} \to a$. Then $\{b_n\} \to b$ if $\exists c \geq 0, \exists N_1 > 0, \forall n \geq N_1, |b_n - b| < c|a_n - a|$.

Proof

Fix $\varepsilon > 0$.

$$\exists N_2, n \geq N_2 \to |a_n - a| < \frac{\varepsilon}{C}$$

Let $N = \max\{N_1, N_2\}$. Then if $n \geq N$:

$$|b_n - b| < C|a_n - a| < C\frac{\varepsilon}{C} = \varepsilon$$

Addition of Sequences

If $\{a_n\} \to a$ and $\{b_n\} \to b$, then $\{a_n + b_n\} \to a + b$.

Proof

Fix $\varepsilon > 0$.

$\exists N_1, N_2, n \geq N_1 \to |a_n - a| < \frac{\varepsilon}{2}, n \geq N_2 \to |b_n - b| < \frac{\varepsilon}{2}$.

Then, let $N = \max\{N_1, N_2\}$. Therefore:

$$n \geq N \to |a_n - a| < \frac{\varepsilon}{2} \wedge |b_n - b| < \frac{\varepsilon}{2}$$

$$n \geq N \to |(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Multiply Sequence by Constant

If $\{a_n\} \to a$ and $\alpha \in \mathbb{R}$, then $\{\alpha a_n\} \to \alpha a$.

Proof

Fix $\varepsilon > 0$. $\forall n, |\alpha a_n - \alpha a| = |\alpha||a_n - a| < 2|\alpha||a_n - a|$

Apply the comparison lemma.

Multiply Zero-Valued Sequence by Zero-Valued Sequence

If $\{a_n\} \to 0$ and $\{b_n\} \to 0$, then $\{a_n b_n\} \to 0$.

Proof

Fix $\varepsilon > 0$. $\exists N_1, N_2, n \geq N_1 \to |a_n - 0| < \sqrt{\varepsilon}, n \geq N_2 \to |b_n - 0| < \sqrt{\varepsilon}$.

Let $N = \max\{N_1, N_2\}$.

$$n \geq N \to |a_n b_n - 0| = |a_n b_n| < \sqrt{\varepsilon}\sqrt{\varepsilon} = \varepsilon$$

Multiply Sequence by Sequence

If $\{a_n\} \to a$ and $\{b_n\} \to b$, then $\{a_n b_n\} \to ab$.

Proof

Let $\alpha_n = a_n - a$, $\beta_n = b_n - b$. Then $\{\alpha_n\} \to 0$, $\{\beta_n\} \to 0$.

Also, $|a_n b_n - ab| = |(\alpha_n + a)(\mathrm{B}_n + b) + ab| = |\alpha_n \beta_n + a\beta_n + b\alpha_n|$.

Observe that $\{\alpha_n \beta_n\} \to 0$, $\{a\beta_n\} \to 0$, $\{b\alpha_n\} \to 0$.

Let $c_n = \{\alpha_n \beta_n + a\beta_n + b\alpha_n\}$. Then, $\{c_n\} \to 0$ and so $\{a_n b_n - ab\} \to 0$, and so $\{a_n b_n\} \to ab$.

Reciprocal of sequence

If $\{b_n\} \to b$, and $b \neq 0$, then $\left\{\frac{1}{b_n}\right\} \to \frac{1}{b}$.

Proof

We must find $C, N_1 > 0$, so that if $n \geq N_1$, $\left|\frac{1}{b_n} - \frac{1}{b}\right| \leq C|b - b_n|$.

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \frac{1}{|b_n||b|}|b - b_n|$$

Therefore, we need to show that $\left\{\frac{1}{|b_n|}\right\}$ is bounded.

Observe that $|b| = |b - b_n + b_n| \leq |b - b_n| + |b_n|$.

Therefore, $|b_n| \geq |b| - |b - b_n|$.

Let $\varepsilon = \left|\frac{b}{2}\right|$.

Then, $\exists N > 0$ such that if $n \geq N_1$, then $|b_n - b| < \varepsilon = \frac{|b|}{2}$

Then, $|b_n| \geq |b| - |b - b_n| \geq |b| - \frac{|b|}{2} = \frac{|b|}{2}$.

Then, $\frac{1}{|b_n|} \leq \frac{2}{|b|}$.

Therefore:

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \frac{1}{|b_n||b|}|b_n - b| \leq \frac{2}{|b|^2}|b_n - b|$$

And therefore, by the comparison lemma, $\left\{\frac{1}{b_n}\right\} \to \frac{1}{b}$.

Division of Sequences

If $\{a_n\} \to a$, and $\{b_n\} \to b, b \neq 0$, then $\left\{\frac{a_n}{b_n}\right\} \to \frac{a}{b}$.

Proof

Let $c_n = \frac{1}{b_n}$. Then, $\left\{\frac{a_n}{b_n}\right\} = \{a_n c_n\}$.

By a proposition, $\{c_n\} \to \frac{1}{b}$, and so $\{a_n c_n\} \to \frac{a}{b}$ by a proposition.

Continuous Functions

A subset of the reals is dense based on sequential denseness

$S \subset \mathbb{R}$ is dense iff, $\forall x \in \mathbb{R}, \exists \{a_n\} \subset S, \{a_n\} \to x$.

Proof

Suppose $S \subset \mathbb{R}$ is dense.

Let $x \in \mathbb{R}$. Let $a_n \in S \cap \left(x - \frac{1}{n}, x + \frac{1}{n}\right)$.

Fix $\varepsilon > 0$. Let $N > \frac{1}{\varepsilon}$.

Then if $n \geq N$,

$$a_n \in \left(x - \frac{1}{n}, x + \frac{1}{n}\right)$$
$$x - \frac{1}{n} < a_n < x + \frac{1}{n}$$
$$-\frac{1}{n} < a_n - x < \frac{1}{n}$$
$$|a_n - x| < \frac{1}{n} \leq \frac{1}{N} < \varepsilon$$

For the alternate direction, suppose that $\forall x \in \mathbb{R}, \exists \{a_n\} \subset S$, $\{a_n\} \to x$. Let $(a, b) \subset \mathbb{R}$. Let $x \in \frac{a+b}{2}$. Let $\varepsilon = \frac{b-a}{2}$.

$\exists \{a_n\} \to x, \exists N > 0, n \geq N \to |a_n - x| < \varepsilon$,

So:

$$-\varepsilon < a_n - x < \varepsilon$$
$$x - \varepsilon < a_n < x + \varepsilon$$
$$a_n \in (x - \varepsilon, x + \varepsilon)$$
$$a_n \in (a, b)$$

Since $a_n \in S$, and $a_n \in (a, b)$, $a_n \in S \cap (a, b)$, and so S is dense.

Nonnegative sequences converge to a nonnegative number

If $a_n \geq 0$ and $\{a_n\} \to a$, then $a \geq 0$.

Proof

Suppose $a < 0$ and let $\varepsilon = \frac{|a|}{2}$. Then $\exists N, n \geq N \to |a_n - a| < \varepsilon = \frac{|a|}{2}$, so

$$-\frac{|a|}{2} < a_n - a < \frac{|a|}{2}$$
$$\frac{a}{2} < a_n - a < -\frac{a}{2}$$
$$\frac{3a}{2} < a_n < \frac{a}{2} < 0$$
$$a_n < 0$$

But a_n was supposed to have the property $a_n \geq 0$. Therefore, the lemma holds.

Squeeze Theoremish Proposition

Suppose $\{a_n\} \subset [b, c]$ and $\{a_n\} \to a$. Then, $a \subset [b, c]$

Proof

Since $a_n \geq b$, $a_n - b \geq 0$,

$$\{a_n - b\} \to a - b \geq 0$$

Similarly, $c - a \geq 0$, so:

$$b \leq a \leq c$$
$$a \in [b, c]$$

Closed Sets

A set $A \subset \mathbb{R}$ is closed if whenever a sequence $\{a_n\} \subset A$ converges to a, then $a \in A$.

Example

If A and B are closed, $A \cup B$ is closed.

Example

$$A_n = \left[-1 + \frac{1}{n}, 1 - \frac{1}{n}\right]$$
$$\bigcup_{n \in \mathbb{N}} A_n = (-1, 1)$$

Open Set

A set $A \in \mathbb{R}$ is open if $\forall x \in A$, $\exists \varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subset A$

Example

$\varnothing$ and $\mathbb{R}$ are open and closed.

Relating Closed and Open Sets

$A \subset \mathbb{R}$ is open iff $\mathbb{R} \setminus A$ is closed.

Go to ∞

$\forall M > 0$, $\exists N > 0$ such that if $n \geq N$, $a_n > M$

Example
$\{n \cdot (-1)^n\}$ does not go to ∞ and is also not bounded.

Monotone

A sequence $\{a_n\}$ is monotone if $\forall n \in \mathbb{N}$, $\{a_{n+1}\} \geq a_n$ or $\forall n \in \mathbb{N}, a_{n+1} \leq a_n$. The first is monotone increasing, and the latter is monotone decreasing.

Monotone Convergence Theorem

Let $\{a_n\}$ be a monotone sequence. $\{a_n\}$ converges iff it is bounded.

If $\{a_n\}$ is monotone increasing then $\{a_n\} \to \sup\{a_n\}$. If $\{a_n\}$ is monotone decreasing, then $\{a_n\} \to \inf\{a_n\}$.

Proof
$(\to)$ Suppose $\{a_n\}$ is monotone and converges. Since $\{a_n\}$ converges, by a theorem it is bounded.

$(\leftarrow)$ Suppose $\{a_n\}$ is monotone increasing and bounded. Because $\{a_n\}$ is bounded, the supremum exists. Let $a = \sup\{a_n\}$.

Fix $\varepsilon > 0$.

Then, there exists N such that $a - \varepsilon < a_N \leq a$.

Since it is monotone increasing, $\forall n \geq N, a_n \geq a_N > a - \varepsilon$. Furthermore, $a_n \leq a$.

Thus, $|a_n - a| = a - a_n \leq a - a_n < \varepsilon$.

Therefore $\{a_n\} \to \sup\{a_n\}$

Similarly if $\{a_n\}$ is monotone decreasing, $\{a_n\} \to \inf\{a_n\}$.

Proposition
The sequence $\left\{\sum_{k=1}^{n} \frac{1}{k}\frac{1}{2^k}\right\}$.

Proof
Let $a_n = \sum_{k=1}^{n} \frac{1}{k}\frac{1}{2^k}$. Since $\forall n \in \mathbb{N}, a_{n+1} - a_n = \frac{1}{n+1}\frac{1}{2^{n+1}} > 0$, $a_n + 1 > a_n$

Therefore it is monotone increasing.

$$0 \leq a_n \leq \sum_{k=1}^{n} \frac{1}{2^k} = \frac{1 - \left(\frac{1}{2}\right)^{n+1}}{1 - \frac{1}{2}} < \frac{1}{1 - \frac{1}{2}} = 2$$

Therefore, $\{a_n\}$ is bounded.

Since it is bounded and monotone increasing, the sequence converges.

Proposition
Let $a_n = \sum_{k=1}^{n} \frac{1}{k}$. $\{a_n\}$ does not converge.

Proof
$a_{n+1} - a_n = \frac{1}{n+1} > 0$, so $a_{n+1} > a_n$, and $\{a_n\}$ is monotone increasing.

Claim: $\forall n \in \mathbb{N}, a_{2^n} \geq 1 + \frac{n}{2}$.

Base case ($n = 2$):

$$a_2 = 1 + \frac{1}{2} = \frac{3}{2}$$

Suppose $a_{2^n} \geq 1 + \frac{n}{2}$.

Then $a_{2^{n+1}} = a_2$

$$\begin{aligned}
a_{2^{n+1}} &= a_{2^n} + \frac{1}{2^n + 1} + \frac{1}{2^n + 2} + \dots + \frac{1}{2^n + 2^n} \\
&\geq a_{2^n} + \frac{1}{2^n + 2^n} + \frac{1}{2^n + 2^n} + \dots + \frac{1}{2^n + 2^n} \\
&\geq a_{2^n} + \frac{2^n}{2 \cdot 2^n} \\
&= a_{2^n} + \frac{1}{2} \\
&\geq 1 + \frac{n}{2} + \frac{1}{2} \\
&\geq 1 + \frac{n+1}{2}
\end{aligned}$$

Thus, $\{a_n\}$ is not bounded and therefore does not converge.

Nested Interval Theorem

Suppose $A_n = [a_n, b_n]$ for $-\infty < a_n < b_n < \infty$. Suppose that $\forall n \in \mathbb{N}, A_{n+1} \subset A_n$. Then, $\bigcap_{n=1}^{\infty} A_n = [\sup\{a_n\}, \inf\{b_n\}] \neq \varnothing$.

Proof

$\{a_n\}$ is monotone increasing and bounded: $a_1 \leq a_n \leq b_1$. Therefore, it converges to $a = \sup\{a_n\}$. Similarly, $\{b_n\} \to b = \inf\{b_n\}$.

Claim: $\bigcap_{n=1}^{\infty} A_n = [\inf\{b_n\}, \sup\{a_n\}]$ and $a \leq b$.

By a homework problem, $a \leq b$. Let $x \in [a, b]$. $\forall n \in \mathbb{N}, x \geq a \geq a_n \land x \leq b \leq b_n$. Therefore, $\forall n \in \mathbb{N}, x \in A_n$, and so $x \in \bigcap_{n=1}^{\infty} A_n$, and so therefore $[a, b] \subset \bigcap_{n=1}^{\infty} A_n$.

Let $x \in \bigcap_{n=1}^{\infty} A_n$.

Therefore, $\forall n \in \mathbb{N}$:

$$x \in A_n$$
$$a_n \leq x \leq b_n$$

x is an upper bound of $\{a_n\}$ and a lower bound of $\{b_n\}$, and thisi $x \geq \sup\{a_n\} = a$ and $x \leq \inf\{b_n\} = b$. So $x \in [a, b]$.

Subsequence

Let $\{n_k\}$ be a sequence of natural numbers that is strictly increasing ($n_1 < n_2 < n_3 < \cdots$).

A subsequence of a sequence $\{a_n\}$ is $\{b_k\} = \left\{a_{n_k}\right\}$.

This is usually just denoted by $\{a_{n_k}\}$.

Example

$\{a_n\} = \{\frac{1}{n}\}$

Let $n_k = k^2$, and then $\{a_{n_k}\} = \{1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, ...\}$.

Subsequences of a Convergent Sequence Converge

If $\{a_n\}$ is a sequence that converges to a, then every subsequence also converges to a.

Proof

Fix $\varepsilon > 0$.

$\exists N_1$ such that $n \geq N_1 \to |a_n - a| < \varepsilon$.

Let $\{a_{n_k}\}$ be a subsequence of $\{a_n\}$. Then, $\{n_k\}$ is a strictly increasing sequence of natural numbers.

Therefore, $\exists N, k \geq N \to n_k \geq N_1$. Thus, if $k \geq N$, $\left|a_{n_k} - a\right| < \varepsilon$.

Peak Index

$m \in \mathbb{N}$ is a peak index of a sequence $\{a_n\}$ if $\forall n \geq m, a_m \geq a_n$.

This means that everything in the future is smaller (or the same).

Every sequence has a monotone subsequence.

Every sequence has a monotone subsequence.

Proof

$\{a_n\}$ has either finitely many peak indices or infinitely many peak indices.

Case 1: Finite Peak Indices

This means that there is $N \in \mathbb{N}$ so that there are no peak indices greater than N.

Let $n_1 = N + 1$. For all k, let n_{k+1} be an integer such that $a_{n_{k+1}} > a_{n_k}$. This is possible because there are no peak indices after N, which means that there is always a bigger point in the future.

Then $\{a_{n_k}\}$ is a monotone increasing subsequence.

Case 2: Infinitely Many Peak Indices

Let n_k be the increasing sequence of peak indices. Then, $\{a_{n_k}\}$ is monotone decreasing.

Bounded Sequences have convergent subsequences

Every bounded sequence has a convergent subsequence.

Proof

Take a monotone subsequence of the sequence, which is possible by above. Since the sequence is bounded, the subsequence is bounded, and so since the subsequence is monotone and bounded, it is convergent.

Sequentially Compact

A set S is sequentially compact if every sequence in S has a subsequence that converges in S.

$S \subset \mathbb{R}, \forall \{a_n\} \subset S, \exists \{n_k\}, \left\{a_{n_k}\right\} \to a, a \in S \leftrightarrow S \text{ is sequentially compact.}$

Example

$[0, 1]$

Let $\{a_n\} \subset [0, 1]$. Then $\{a_n\}$ is bounded, and therefore there is a subsequence that converges. Call this $\left\{a_{n_k}\right\} \to a$. Since $[0, 1]$ is closed, any sequence within converges within, and so $a \in [0, 1]$, and therefore $[0, 1]$ is sequentially compact.

Non-Examples

- $(0, 1)$
  - $\left\{\frac{1}{2n}\right\} \to 0, 0 \notin (0, 1)$
- $\mathbb{R}$
  - $\{n\}$ does not converge.

Bolzano-Weierstrass Theorem

A set S is sequentially compact iff it is closed and bounded.

Proof

$(\to)$ Let S be sequentially compact set.

Let $\{a_n\} \subset S$, so that $\{a_n\} \to a \in \mathbb{R}$. $\exists \left\{a_{n_k}\right\} \to b \in S$. But since $\{a_n\} \to a$, $a = b, b \in S \to a \in S$, and therefore S is closed.

Assume S is not bounded.

$$\forall n \in \mathbb{N}, a_n \in S, |a_n| \geq n$$

Let $\left\{a_{n_k}\right\}$ be a subsequence of $\{a_n\}$.

Let $M \geq 0$. Then if $n \geq M$, $|a_n| \geq n \geq M$.

Then, if $k \geq M$, $n_k \geq k \geq M$, so $\left|a_{n_k}\right| \geq M$. Thus $\left\{a_{n_k}\right\} \to \infty$ and does not converge.

Therefore, S must be bounded.

$(\leftarrow)$ Let S be a closed and bounded set.

Let $\{a_n\} \subset S$. Then $\{a_n\}$ is bounded, and so a subsequence $\left\{a_{n_k}\right\}$ converges to a. Because S is closed $\left\{a_{n_k}\right\} \subset S \to a \in S$.

Continuous function

A function $f : D \subset \mathbb{R} \to \mathbb{R}$ is continuous if for all $x_0 \in D$ and all $\{x_n\} \subset D$ converging to x_0,

$$\lim_{n \to \infty} f(x_n) = f(x_0)$$

Examples

$$f(x) = 3x^2 + 2x - 1$$
$$f : \mathbb{R} \to \mathbb{R}$$

This is a continuous function. Let $x_0 \in \mathbb{R}$. Let $\{x_n\} \to x_0$. Then $\{f(x_n)\} = \{3x_n^2 + 2x_n - 1\} = 3\{x_n\}^2 + 2\{x_n\} - \{1\} \to x_0$ by various theorems about manipulating sequences.

$$f(x) = \begin{cases} 1 & x < 0 \\ 2 & x \geq 0 \end{cases}, f : \mathbb{R} \to \mathbb{R}$$

Let $\{x_n\} = \{-\frac{1}{n}\}$. Then $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f\left(-\frac{1}{n}\right) = 1$ but $f(0) = 2$, and therefore $f(x)$ is discontinuous.

$$f(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & x \notin \mathbb{Q} \end{cases}$$

This function is discontinuous everywhere.

Manipulating Continuous Functions

Suppose $f, g : D \to \mathbb{R}$ are both continuous. Then:

- $f + g$ is continuous
- $f - g$ is continuous
- $f \times g$ is continuous
- $g(x) \neq 0 \to \frac{f}{g}$ is continuous

Composing Continous Functions

Let $f : U \to D$ and $g : D \to \mathbb{R}$ be continuous. Then $g \circ f$ is also continuous.

Proof

Let $\{x_n\} \subset U$ with $\{x_n\} \to x_0 \in U$. Let $y_n = f(x_n)$. Since f is continuous, $\{f(x_n)\} = \{y_n\} \to f(x_0) := y_0$. Since $f : U \to D$, $\{y_n\} \subset D$ and $y_0 \in D$. Since g is continuous, $\{g(y_n)\} \to g(y_0)$. Thus, $\{g\{f(x_n)\}\} = \{g(y_n)\} \to g(y_0) = g(f(x_0))$, and therefore $g \circ f$ is continuous.

Maximum and Maximizer

Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be continuous. If there $\exists x_o \in D, \forall x \in D, f(x_0) \geq f(x)$, then x_0 is a maximizer of f and $f(x_0)$ is the maximum value.

Examples

- $f(x) = 1, D = \mathbb{R}$
  - maximizers are $\mathbb{R}$
- $f(x) = -x^2, D = \mathbb{R}$
  - maximizers are $\{0\}$, and a max value of 0.
- $f(x) = x, D = (0, 1)$
  - no maximizer
- $f(x) = x, D = \mathbb{R}$
  - no maximizer

Minimum and Minimizer

Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be continuous. If there $\exists x_o \in D, \forall x \in D, f(x_0) \leq f(x)$, then x_0 is a minimizer of f and $f(x_0)$ is the minimum value.

Image

Let $f : D \subset A \to B$, $f(D) = \{f(x) \mid x \in D\}$.

Image of a sequentially compact set is sequentially compact

If D is sequentially compact and $f : D \to \mathbb{R}$ is continuous, then $f(D)$ is sequentially compact.

Proof

Let $\{y_n\} \subset f(D)$.

For each n, there exists $x_n \in D$ such that $y_n = f(x_n)$, by definition of image.

Since D is sequentially compact, there exists a subsequence $\left\{x_{n_k}\right\}$ that converges to $x_0 \in D$. This can create a subsequence $\left\{f\left(x_{n_k}\right)\right\} = \left\{y_{n_k}\right\}$.

Since f is continuous, $\lim_{k\to\infty} f\left(x_{n_k}\right) = \lim_{k\to\infty} y_{n_k} = f(x_0) := y_0$. Therefore, there exists a subsequence of $\{y_n\}$ that converges in D, and so $f(D)$ is sequentially compact.

Extreme Value Theorem

If $f : D \to \mathbb{R}$ is continuous, and D is sequentially compact, and f has both a maximizer and a minimizer in D or f attains both its max and minimum values.

Proof

Since D is sequentially compact and f is continuous, $f(D)$ is sequentially compact. We will show that $f(D)$ has a maximum.

Since $f(D)$ is sequentially compact, it is closed and bounded. Therefore, $\sup f(D)$ exists.

$\forall n \in \mathbb{N}$, let $a_n \in f(D)$ such that $\sup f(D) - \frac{1}{n} < a_n \leq \sup f(D)$. Fix $\varepsilon > 0$. Let $N > \frac{1}{\varepsilon}$. Then if $n \geq N$,

$$0 \leq \sup f(D) - a_n \leq \frac{1}{n} \leq \frac{1}{N} < \varepsilon$$
$$\Rightarrow |\sup f(D) - a_n| < \varepsilon$$

Since $\{a_n\} \to \sup f(D)$ and $f(D)$ is closed, $\sup f(D) \in f(D)$ and thus $f(D)$ contains a maximum value $(\sup f(D))$.

Similarly, $f(D)$ has a minimum value.

Intermediate Value Theorem

If f is continuous on the interval $[a, b]$, and $f(a) < c < f(b)$ or $f(b) < c < f(a)$, then $\exists x \in [a, b], f(x) = c$.

Proof

Without loss of generality, $f(a) < f(b)$. Let $f(a) < c < f(b)$.

Let $a_1 = a$, $b_1 = b$. For $n \in \mathbb{N}$, let $m_n = \frac{a_n + b_n}{2}$. If $f(m_n) \leq c$ let $a_{n+1} = m_n, b_{n+1} = b_n$. Otherwise, if $f(m_n) > c$, let $a_{n+1} = a_n, b_{n+1} = m_n$.

We claim that

$$\forall n, a \leq a_n \leq a_{n+1} \leq b_{n+1} \leq b_n \leq b$$

Base step: $b_1 \geq a_1$.

Inductive step:

If $b_n \geq a_n$, then either $b_{n+1} = b_n$ and $a_{n+1} = \frac{a_n+b_n}{2} < \frac{b_n+b_n}{2} \leq b_{n+1}$ or $a_{n+1} = a_n$ and $b_{n+1} = \frac{a_n+b_n}{2} \geq \frac{a_n+a_n}{2} = a_{n+1}$.

Outside of induction, if $a_{n+1} = a_n$, then $a_n < a_{n+1}$. If $a_{n+1} = m_n = \frac{a_n+b_n}{2} \geq \frac{a_n+a_n}{2} = a_n$.

This is similarly the case for b_n.

$\{a_n\}$ and $\{b_n\}$ are monotone and bounded, so $\{a_n\} \to a_0$ and $\{b_n\} \to b_0$.

We claim that $\forall n, b_n \geq a_0$.

Suppose $b_n < a_0$. Then, there exists m such that $b_m < a_n \leq a_0$.

Without loss of generality, $m \geq n$. Since the sequences are monotone, $b_m \leq b_n \leq a_m \leq a_0$, and therefore $b_n \geq a_0$. Similarily, $\forall n, a_n \leq b_0$.

Consider $\{b_n - a_n\} \to b_0 - a_0$.

$$b_{n+1} - a_{n+1} = \begin{cases} \frac{a_n+b_n}{2} - a_n \\ b_n - \frac{a_n+b_n}{2} \end{cases} = \frac{b_n - a_n}{2}$$

By induction, $b_n - a_n = \frac{b-a}{2^n}$. $\left\{\frac{b-a}{2^n}\right\} \to 0$, and therefore $0 = b_0 - a_0$. By continuity, $\lim_{n\to\infty} f(a_n) = f(a_0)$ and $\lim_{n\to\infty} f(b_n) = f(b_0)$. By construction, $f(a_n) \leq c$, and $f(b_n) \geq c$. Therefore, $f(a_0) \leq c$, and $f(b_0) \geq c$.

$c \leq f(b_0) = f(a_0) \leq c \to f(b_0) = f(a_0) = c$

Roots exist

$\forall c > 0, m \in \mathbb{N}, \exists x \in \mathbb{R}, x^m = c$

Proof

$$f(x) = x^m$$

Note $f(0) = 0$ and therefore $0 < c$. Further note $f(c+1) = c^m + ... + mc + 1 > c$ because $m \in \mathbb{N}$ and therefore $m \geq 1$.

By the intermediate value theorem, there exists $x \in [0, c+1]$ such that $x^m = c$.

Image of an interval is an interval

If I is an interval and f is continuous, then $f(I)$ is an interval.

Proof

Case 1: $I = [a, b]$. Since I is sequentially compact, we can let $\alpha = \min(f(I)), \beta = \max(f(I))$. We claim that $f(I) = [\alpha, \beta]$.

Let $f(x_1) = \alpha, f(x_2) = \beta$. WLOG, assume $x_1 \leq x_2$. Let $\alpha < c < \beta$. Then $\exists x \in [x_1, x_2], [x_1, x_2] \subset [a, b]$ such that $f(x) = c$. Thus, $c \in f(I)$ and $f(I) = [\alpha, \beta]$.

Uniform Continuity

A function $f : D \to \mathbb{R}$ is uniformly continuous if for all sequences $\{a_n\}, \{b_n\} \subset D$ if $\lim_{n\to\infty}(a_n - b_n) = 0$ then $\lim_{n\to\infty}(f(u_n) - f(v_n))$.

Example

$f(x) = x^2$ is not uniformly continuous.

Let $\{a_n\} = n$ and $\{b_n\} = \left\{n + \frac{1}{n}\right\}$. Then, $\{a_n - b_n\} = \left\{-\frac{1}{n}\right\} \to 0$.

But $\{f(a_n) - f(b_n)\} = \left\{n^2 - \left(n - \frac{1}{n}\right)^2\right\} = \left\{n^2 - n^2 - 2 - \frac{1}{n^2}\right\} = \left\{-2 - \frac{1}{n^2}\right\} \to -2$

Intuitively, the slope increases too fast.

Example

$f(x) = \frac{1}{x}$ is not uniformly continuous on $(0, 1)$.

Continuous from uniform continuity

Every uniformly continuous function is continuous.

Proof

Let $\{b_n\} = \{x_0\} \subset D$.

Then, $\{a_n\} \to x_0 \leftrightarrow \{a_n - b_n\} \to 0$.

Then, $\{f(a_n) - f(b_n)\} \to 0$, which is equal to $\{f(a_n) - f(x_0)\} \to 0$, so $\{f(a_n)\} \to f(x_0)$, which is equivalent to the definition of continousity.

Uniform continuity from Continuity

Suppose $f : D \to \mathbb{R}$ is continuous and D is sequentially compact, then f is uniformly continuous.

Proof

Suppose f is not uniformly continuous. Then there exists $\{a_n\}, \{b_n\} \subset D$ such that $\{a_n - b_n\} \to 0$ but $\{f(a_n) - f(b_n)\} \not\to 0$.

Then there exists $\varepsilon > 0$ and subsequences also called $\{a_n\}, \{b_n\}$ such that $|f(a_n) - f(b_n)| > \varepsilon$ for all n.

By sequential compactness, there exists $\left\{a_{n_k}\right\}, \left\{b_{n_k}\right\}$[2] such that $\left\{a_{n_k}\right\} \to a$ and $\left\{b_{n_k}\right\} \to b$, $a, b \in D$. Since $\{a_n - b_n\} \to 0$, $a = b = x_0 \in D$.

Then $\left\{f\left(a_{n_k}\right)\right\} \to \{f(x_0)\}$ and $\left\{f\left(b_{n_k}\right)\right\} \to \{f(x_0)\}$. Then, $\left\{f\left(a_{n_k}\right) - f\left(b_{n_k}\right)\right\} \to f(x_0) - f(x_0) = 0$, but $0 < \varepsilon$, a contradiction.

Epsilon-Delta Criterion

$f : D \to \mathbb{R}$ satisfies the $\varepsilon - \delta$ criterion at $x_0 \in D$ if $\forall \varepsilon > 0, \exists \delta > 0, |x - x_0| < \delta \land x \in D \to |f(x) - f(x_0)| < \varepsilon$.

[2]This is questionable, but it does work by first finding a $\left\{a_{n_k}\right\} \to a$, and then finding a subsequence of $\left\{b_{n_k}\right\}$, such that $\left\{b_{n_{k_k}}\right\} \to b$. Since $\left\{a_{n_k}\right\} \to a$, $\left\{a_{n_{k_k}}\right\} \to a$, and so these subsequences do exist

Example
Show that $f(x) = x^3$ satisfies the $\varepsilon - \delta$ criterion at $x_0 = 2$.

Scratch

$$\begin{aligned} & |(x-2)| \\ = & \left|(x-2)(x^2+2x+4)\right| \\ = & \underbrace{\left|x^2+2x+4\right|}_{\text{not huge}}\underbrace{|x-2|}_{\text{small}} \end{aligned}$$

Assume $|x-2| < 1$. Then,

$$\leq 19|x-2|$$

Proof
Fix $\varepsilon > 0$. Choose $\delta < \min\left\{1, \frac{\varepsilon}{19}\right\}$.

Since $|x - x_0| < \delta$, then $|x - x_0| < 1$ and then by above, $\left|x^3 - 2^3\right| < 19\delta < 19\frac{\varepsilon}{19} = \varepsilon$

Relating epsilon-delta criterion and continuity

Let $f : D \to \mathbb{R}$ be continuous iff it satisfies the epsilon-delta criterion at all $x_0 \in D$.

Uniformly Continuity by $\varepsilon - \delta$ criterion

$f : D \to \mathbb{R}$ is uniformly continuous iff $\forall \varepsilon > 0, \exists \delta > 0$ such that $\forall u, v \in D$, if $|u - v| < \delta$ then $|f(u) - f(v)| < \varepsilon$.

This makes sense because uniformness means that the δ only depends on ε.

Example
Prove that $f(x) = x^2$ is continuous at $x = x_0$ using the $\varepsilon - \delta$ criterion.

Fix $\varepsilon > 0$.

Scratch
If $|x - x_0| < \delta$ Then

$$\begin{aligned} |f(x) - f(x_0)| &= \left|x^2 - x_0^2\right| \\ &= |x - x_0||x + x_0| \\ &< \delta(|x| + |x_0|) \\ &< \delta(2|x_0| + \delta) \\ &< \delta(2|x_0| + 1) \end{aligned}$$

$$\delta(2|x_0| + 1) = \varepsilon \to \delta = \min\left(\frac{\varepsilon}{2|x_0| + 1}, 1\right)$$

Work
Let $\delta = \min\left(\frac{\varepsilon}{2|x_0|+1}, 1\right)$.

$$|x - x_0| < \delta$$
$$|f(x) - f(x_0)| \leq |x - x_0||x| + f(x_0)| < \delta(2|x_0| + 1) < \varepsilon$$

This would *not* work for uniform continuity since δ depends on x_0.

Monotone Function

$f : D \to \mathbb{R}$ is monotone increasing if $\forall a, b \in D$ such that $a \leq b$, $f(a) \leq f(b)$ or is monotone decreasing if $\forall a, b \in D$ such that $a \leq b$, $f(a) \geq f(b)$.

Continuity by image and monotone

If $f : D \to \mathbb{R}$ is monotone such that $f(D)$ is an interval, then f is continuous.

Proof

Suppose $f(D) = I$, but f is not continuous.

Then, there exists a sequence $\{x_n\} \to x_0$ such that $\{f(x_n)\} \nrightarrow f(x_0)$.

Let $y_n = f(x_n)$, $y_0 = f(x_0)$. Then there exists ε and a subsequence y_{n_k} so that $\forall k, \left|y_{n_k} - y_0\right| \geq \varepsilon$.

WLOG, assume f is monotone increasing.

Then, there are subsequences we will also call $\left\{y_{n_k}\right\}$ and $\left\{x_{n_k}\right\}$ such that $\left\{x_{n_k}\right\}$ is monotone. WLOG, assume $\left\{x_{n_k}\right\}$ is monotone increasing.

Since $f\left(x_{n_k}\right) = y_{n_k} < y_0 - \varepsilon$ and $f(x_0) = y_0$, there must exist a $x^* \in (x_n, x_0)$ such that $f(x^*) = y_0 - \varepsilon$.

Since $\left\{x_{n_k}\right\} \to x_0$, there is a k such that $x^* < x_{n_k} < x_0$ but $f\left(x_{n_k}\right) = y_{n_k}$ and $y_{n_k} < y_0 - \varepsilon$, so this shouldn't be true and therefore there is a contradiction.

Strictly Monotone Function

A function $f : D \to \mathbb{R}$ is strictly monotone increasing if whenever $x, y \in D$ such that $x < y$, then $f(x) < f(y)$. For strictly decreasing, $x < y$ implies $f(x) > f(y)$.

Put simply, monotone but cannot go sideways.

Strictly monotone functions are injective

A strictly monotone function is injective.

Proof

Let $f : D \to \mathbb{R}$ be strictly monotone. WLOG, assume it is increasing.

Let $x, y \in D$ such that $f(x) = f(y)$. On the contrary, assume $x \neq y$. WLOG, assume $x < y$.

Since $x < y$, $f(x) < f(y)$, but $f(x) = f(y)$ so this is impossible and therefore a contradiction.

Therefore $x = y$, and f is injective.

Strictly monotone functions can be bijective

If f is strictly monotone, then $f : D \to f(D)$ is bijective.

Strictly monotone functions can have an inverse

A strictly monotone function $f : D \to f(D)$ has an inverse f^{-1}.

Inverses are continuous

If D is an interval and $f : D \to f(D)$ is strictly monotone, then $f^{-1} : f(D) \to D$ is continuous.

Proof

Firstly, f^{-1} exists by above, and $f^{-1}(f(D)) = D$.

Let $a, b \in f(D)$ such that $a \leq b$. WLOG, assume f is monotone increasing.

Let $x = f^{-1}(a)$ and $y = f^{-1}(b)$. Therefore, $a = f(x)$ and $b = f(y)$. Therefore, $a \leq b$ and so $f(x) \leq f(y)$ and therefore $x \leq y$.

Thus $f^{-1}(a) < f^{-1}(b)$. Since f^{-1} is monotone and $f^{-1}(f(D))$ is an inverval, f^{-1} is continuous.

$x^{\frac{m}{n}}$ exists and define x^r

Let $n \in \mathbb{N}$ and $f : [0, \infty) \to \mathbb{R}$ be defined by $f(x) = x^n$. Then, f is strictly monotone increasing.

f^{-1} exists and is denoted $f^{-1}(x) = x^{\frac{1}{n}}$. It is continuous. Then, $x^{\frac{m}{n}} = \left(x^{\frac{1}{n}}\right)^m$.

Let $r \in (0, \infty)$.

Let $\{a_n\} \subset \mathbb{Q}$ such that $\{a_n\} \to r$. Define $x^r = \lim_{n \to \infty} x^{a_n}$.[3]

Limit Point

Let $D \subset \mathbb{R}$. $x_0 \in \mathbb{R}$ is a limit point of D if $\exists \{x_n\} \subset D \setminus \{x_0\}$ such that $\{x_n\} \to x_0$.

Limit

$f : D \to \mathbb{R}$ has a limit L as x approaches a limit point x_0 if for all sequences $\{x_n\} \subset D \setminus \{x_0\}$ such that $\{x_n\} \to x_0$, $\{f(x_n)\} \to L$.

This is written:

$$\lim_{x \to x_0} f(x) = L$$

Algebraic Limit Theorem

If $\lim_{x \to x_0} f(x) = L_1, \lim_{x \to x_0} f(x) = L_2$:

$$\lim_{x \to x_0} f(x) + g(x) = L_1 + L_2$$

$$\lim_{x \to x_0} f(x) - g(x) = L_1 - L_2$$

$$\lim_{x \to x_0} f(x)g(x) = L_1 L_2$$

$$L_2 \neq 0 \to \lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{L_1}{L_2}$$

Continuous Functions and Limits

Iff $f : D \to \mathbb{R}$ is continuous and $x_0 \in D$ and x_0 is a limit point of D

[3] Unproven that this exists but oh well it does.

$$\lim_{x \to x_0} f(x) = f(x_0)$$

Limit Definition of the Derivative

$f : D \to \mathbb{R}$ is differentiable at $x_0 \in D$ if x_0 is a limit point of D and

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists, and $f'(x_0)$ is the derivative.

Example

Find $f'(x)$ for $f(x) = mx + b$.

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{mx + b - mx_0 + b}{x - x_0} = \lim_{x \to x_0} m \frac{x - x_0}{x - x_0} = \lim_{x \to x_0} m = m$$

Therefore $f'(x) = m$.

Example

Find $f'(x)$ for $f(x) = x^n$.

$$\begin{aligned} f'(x_0) &= \lim_{x \to x_0} \frac{x^n - x_0^n}{x - x_0} = \lim_{x \to x_0} \frac{(x - x_0)\left(x^{n-1} + x_0 x^{n-2} + x_0^2 x^{n-3} + \cdots + x_0^{n-2} x + x_0^{n-1}\right)}{x - x_0} \\ &= \lim_{x \to x_0} \left(x^{n-1} + x_0 x^{n-2} + x_0^2 x^{n-3} + \cdots + x_0^{n-2} x + x_0^{n-1}\right) \\ &= x_0^{n-1} + x_0 x_0^{n-2} + x_0^2 x_0^{n-3} + \cdots + x_0^{n-2} x_0 + x_0^{n-1} \\ &= n x_0^{n-1} \end{aligned}$$

Differentiability implies continuity

A differentiable function is continuous.

Proof

Let $f : D \to \mathbb{R}$ be differentiable.

Then, $\forall x_0 \in D$

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \quad \text{exists}$$

Furthermore, $\lim_{x \to x_0} x - x_0 = 0$

Then:

$$\begin{aligned} \lim_{x \to x_0} f(x) - f(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}(x - x_0) &= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \lim_{x \to x_0} (x - x_0) \\ &= \left(\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}\right) 0 = 0 \end{aligned}$$

Since $f(x_0) - f(x_0) = 0$ at each point, $f(x)$ is continuous.

Example: It's not the other way

Show $|x|$ is not differentiable at $x_0 = 0$.

$$\lim_{x\to 0} \frac{|x| - |0|}{x - 0} = \lim_{x\to 0} \frac{|x|}{x}$$

Take the sequences $\{a_n\} = \{1/n\}$ and $\{b_n\} = \{-1/n\}$.

$$\left\{\frac{|a_n|}{a_n}\right\} = \left\{\frac{\left|\frac{1}{n}\right|}{\frac{1}{n}}\right\} = \frac{\frac{1}{n}}{\frac{1}{n}} = 1$$

$$\left\{\frac{|b_n|}{b_n}\right\} = \left\{\frac{\left|-\frac{1}{n}\right|}{-\frac{1}{n}}\right\} = \frac{\frac{1}{n}}{-\frac{1}{n}} = -1$$

But $1 \neq -1$, so the limit does not exist.

Combining Derivatives

If $f, g : D \to \mathbb{R}$ that are differentiable, then:

1. $(f + g)'(x) = f'(x) + g'(x)$
2. $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$
3. $\left(\frac{1}{g}\right)'(x) = -\frac{g'(x)}{(g(x))^2}$, if $g(x) \neq 0$
4. $\left(\frac{f}{g}\right)'(x) = \frac{f'g(x) - f(x)g'(x)}{(g(x))^2}$, if $g(x) \neq 0$

Proof

Let $x_0 \in D$.

$$\begin{aligned}
(f + g)'(x_0) &= \lim_{x\to x_0} \frac{f(x) + g(x) - (f(x_0) + g(x_0))}{x - x_0} \\
&= \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} + \frac{g(x) - g(x_0)}{x - x_0} \\
&= \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} + \lim_{x\to x_0} \frac{g(x) - g(x_0)}{x - x_0} \\
&= f'(x_0) + g'(x_0)
\end{aligned}$$

$$
\begin{aligned}
(fg)'(x_0) &= \lim_{x \to x_0} \frac{f(x)g(x) - f(x_0)g(x_0)}{x - x_0} \\
&= \lim_{x \to x_0} \frac{f(x)g(x) + f(x)g(x_0) - f(x)g(x_0) - f(x_0)g(x_0)}{x - x_0} \\
&= \lim_{x \to x_0} \frac{f(x)g(x) - f(x)g(x_0) + f(x)g(x_0) - f(x_0)g(x_0)}{x - x_0} \\
&= \lim_{x \to x_0} \frac{f(x)g(x) - f(x)g(x_0)}{x - x_0} + \frac{f(x)g(x_0) - f(x_0)g(x_0)}{x - x_0} \\
&= \lim_{x \to x_0} f(x)\frac{g(x) - g(x_0)}{x - x_0} + g(x_0)\frac{f(x) - f(x_0)}{x - x_0} \\
&= \lim_{x \to x_0} f(x) \lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0} + \lim_{x \to x_0} g(x_0) \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \\
&= f(x_0)g'(x) + g(x_0)f'(x_0)
\end{aligned}
$$

Neighborhood

Let $x_0 \in \mathbb{R}$. $I \subset \mathbb{R}$ is a neighborhood of x_0 if I is an open interval x_0.

Change of variables for limits

Let $x_0 \in \mathbb{R}$, I be a neighborhood of x_0, and $f : I \to \mathbb{R}$ be continuous.

Then, if $y_0 = f(x_0)$ and $\lim_{y \to y_0} g(y)$ exists,

$$\lim_{x \to x_0} g(f(x)) = \lim_{y \to y_0} g(y)$$

Proof

Let $J = f(I)$. J is an interval that contains y_0.

Let $\{x_n\} \subset I$, $\{x_n\} \to x_0$.

Let $y_n = f(x_n)$. $\{y_n\} \to y_0$ due to continuity.

We know $\lim_{y \to y_0} g(y) = L$, so $\{g(y_n)\} \to L$.

Furthermore, $\{g(f(x_n))\} \to L$

So $\lim_{x \to x_0} g(f(x)) = L$.

Invertibility for change of variables

If $f : I \to \mathbb{R}$ is continuous and invertible then, $\forall x_0 \in I, y_0 = f(x_0)$, then

$$\lim_{x \to x_0} g(f(x)) = \lim_{x \to x_0} g(y)$$

Derivative of Inverse

Suppose $x_0 \in \mathbb{R}$, I is a neighborhood of x_0 and $f : I \to \mathbb{R}$ is differentiable with $f'(x_0) \neq 0$, then if f is invertible:

$$\left(f^{-1}\right)'(f(x_0)) = \frac{1}{f'(x_0)}$$

Proof

Let $y = f(x)$.

$$\begin{aligned}
\left(f^{-1}\right)'(y_0) &= \lim_{y \to y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} \\
&= \lim_{y \to y_0} \frac{x - x_0}{f(x) - f(x_0)} \\
&= \lim_{x \to x_0} \frac{x - x_0}{f(x) - f(x_0)} \\
&= \frac{1}{\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}} \\
&= \frac{1}{f'(x_0)}
\end{aligned}$$

Derivative of Root

$$\begin{aligned}
f(x) &= x^n \\
f^{-1}(x) &= x^{1/n} \\
\left(f^{-1}\right)'(f(x)) &= \frac{1}{f'(x)} \\
\left(f^{-1}\right)'(y) &= \frac{1}{f'(f^{-1}(y))} \\
\left(f^{-1}\right)'(y) &= \frac{1}{n\left(y^{\frac{1}{n}}\right)^{n-1}} = \frac{1}{n} y^{\frac{1}{n}-1}
\end{aligned}$$

Chain rule

Suppose $f : I \to \mathbb{R}$ and $g : f(I) \to \mathbb{R}$ are differentiable.

Then,

$$(g \circ f)'(x) = g'(f(x)) f'(x)$$

Proof

Let $x_0 \in I$, $y = f(x)$, $y_0 = f(x_0)$.

$$\frac{g(f(x)) - g(f(x_0))}{x - x_0} = \frac{g(y) - g(y_0)}{y - y_0} \cdot \frac{f(x) - f(x_0)}{x - x_0}$$

If a neighborhood I' of x_0, $I' \subset I$ where f is invertible exists, then

$$\begin{aligned}
&\lim_{x\to x_0} \frac{g(f(x))-g(f(x_0))}{x-x_0} \\
&= \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} \lim_{y\to y_0} \frac{g(y)-g(y_0)}{y-y_0} \\
&= f'(x_0)g'(y_0) = f'(x_0)g'(f(x_0))
\end{aligned}$$

If it is not invertible at $x = x_0$, then there $\{y_n\} \to x_0$ and $\{z_n\} \to x_0$, $\{y_n\}, \{z_n\} \subset I$ such that $y_n \neq z_n$ but $f(y_n) = f(z_n)$ for all $n \in \mathbb{N}$.

Then

$$\begin{aligned}
f'(x_0) = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} &= \lim_{n\to\infty} \frac{f(y_n)-f(x_0)}{y_n-x_0} \\
&= \lim_{n\to\infty} \frac{f(z_n)-f(x_0)}{z_n-x_0}
\end{aligned}$$

Power Rule (for rationals)

Let $r \in \mathbb{Q}$, $r \geq 0$. Let $f(x) = x^r$. Then, $f'(x) = rx^{r-1}$.

Proof

Let $r = \frac{m}{n}$. Let $g(x) = x^m$ and $h(x) = x^{\frac{1}{n}}$. Then, $f(x) = g(h(x))$.

By the chain rule, $f'(x) = h'(x)g'(h(x))$.

$g'(x) = mx^{m-1}$, $h'(x) = \frac{1}{n}x^{\frac{1}{n}-1}$

$$\begin{aligned}
f'(x) &= \frac{1}{n}x^{\frac{1}{n}-1} \cdot m\left(x^{\frac{1}{n}}\right)^{m-1} \\
&= \frac{m}{n}x^{\frac{1}{n}-1}x^{\frac{m}{n}-\frac{1}{n}} \\
&= \frac{m}{n}x^{\frac{m}{n}-1} \\
&= rx^{r-1}
\end{aligned}$$

The Derivative is Zero at a Maximzer

Suppose $f : [a, b] \to \mathbb{R}$ is differentiable with maximizer $x_0 \in (a, b)$. Then, $f'(x_0) = 0$.

Proof

$$\begin{aligned}
f'(x_0) = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} &= \lim_{\substack{x \to x_0 \\ x < x_0}} \frac{f(x)-f(x_0)}{x-x_0} \\
&= \lim_{\substack{x \to x_0 \\ x > x_0}} \frac{f(x)-f(x_0)}{x-x_0}
\end{aligned}$$

If $x < x_0$, $f(x) - f(x_0) \leq 0$, and $x - x_0 < 0$, and so therefore:

$$\frac{f(x) - f(x_0)}{x - x_0} \geq 0$$

Therefore:

$$\lim_{\substack{x \to x_0 \\ x < x_0}} \frac{f(x) - f(x_0)}{x - x_0} \geq 0$$

Alternatively, if $x > x_0$. $f(x) - f(x_0) \geq 0$, $x - x_0 > 0$, and so therefore:

$$\frac{f(x) - f(x_0)}{x - x_0} \leq 0$$

Therefore $f'(x_0) = 0$.

Rolle's Theorem

Suppose $f : I \to \mathbb{R}$ is differentiable, and suppose $a, b \in I$ such that $a < b$, and $f(a) = f(b)$. Then, $\exists x_0 \in (a, b)$ such that $f'(x_0) = 0$.

Proof

f is continuous on $[a, b]$. Let $m = \min_{x \in [a,b]} f(x)$, $M = \max_{x \in [a,b]} f(x)$.

Case 1: $m = M$

Then $f(x)$ is constant for all $x \in (a, b)$, so $f'(x) = 0$ for all $x \in (a, b)$.

Case 2: $f(a) = m \vee f(b) = m, m \neq M$

Then, $M = f(x_0)$, $x_0 \in (a, b)$.

Case 3: $f(a) = M \vee f(b) = M, m \neq M.$

Then, the maximum of g, $g = -f$, occurs at $x_0 \in (a, b)$, and $f'(x_0) = -g'(x_0)$.

Case 4: $M \in f(x_0), x_0 \in (a, b)$:

By the lemma, $f'(x_0) = 0$, or in case 3, $g'(x_0) = 0 \Rightarrow f'(x_0) = 0$.

Mean Value Theorem

Let $f : I \to \mathbb{R}$ be differentiable. Let $a, b \in I$, $a < b$.

Then, $\exists x_0 \in (a, b)$, such that $f'(x_0) = \frac{f(b)-f(a)}{b-a}$

Proof

Let $g(x) = f(x) - x\frac{f(b)-f(a)}{b-a}$

$g : I \to \mathbb{R}$ is differentiable and $g(a) = g(b)$.

$$g(a) = f(a) - a\frac{f(b) - f(a)}{b - a} = \frac{f(a)(b - a) + af(b) - af(a)}{b - a} = \frac{bf(a) - af(a) + af(b) - af(a)}{b - a}$$

$$g(b) = f(b) - b\frac{f(b) - f(a)}{b - a} = \frac{f(b)(b - a) + bf(b) - bf(a)}{b - a} = \frac{bf(b) - af(b) + bf(b) - bf(a)}{b - a}$$

By Rolle's theorem, $g'(x_0) = 0$. $f'(x_0) = g'(x_0) + \frac{f(b)-f(a)}{b-a} = \frac{f(b)-f(a)}{b-a}$.

Identity Criterion

$f : I \to \mathbb{R}$ is constant iff $f' \equiv 0$.

Proof

$(\to)$ If $f(x) = \mathbb{c}$, then

$$f'(x) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{\mathbb{c} - \mathbb{c}}{x - x_0} = \lim_{x \to x_0} 0 = 0$$

$(\leftarrow)$ Suppose $f'(x) = 0$. Choose $x_0 \in I$ and let $f(x_0) = \mathbb{c}$.

Let $x \in I$. Then, by the MVT, $\exists x_1 \in (x, x_0)$ such that $f'(x_1) = \frac{f(x_0) - f(x)}{x_0 - x} = 0$.

Thus, $f(x) = f(x_0) = \mathbb{c}$ for all x.

Equal derivatives differ by a constant.

If $f, g : I \to \mathbb{R}$ are differentiable and $f'(x) = g'(x)$ for all $x \in I$, then $\exists c \in \mathbb{R}$ such that $f(x) = g(x) + c$

Proof

Let $h(x) = f(x) - g(x)$. Then $h'(x) = f'(x) - g'(x) = 0$. Then by the lemma, $h(x) = c$ for some $c \in \mathbb{R}$. Thus, $f(x) = g(x) + c$.

Strictly Increasing by derivative

Suppose $f : I \to \mathbb{R}$ is differentiable, then $f(x)$ is strictly monotone increasing if $f'(x) > 0$ for all $x \in I$.

Proof

Suppose $f'(x) > 0$ for all x. Then, let $u, v \in I$, $u < v$. Then, by the MVT, $\exists x_0 \in (u, v)$ such that

$$f'(x_0) = \frac{f(v) - f(u)}{v - u} > 0$$

Thus $f(v) > f(u)$.

Example

$$f(x) = \begin{cases} x^2 \cos\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x == 0 \end{cases}$$

- $f'(0) = 0$
- f is not monotone near `0`
- $f'(x)$ is not continuous.

Example

Show that $1 + x + x^5 = 0$ has exactly one solution.

$f(x) = 1 + x + x^5$ is continuous on the real numbers.

$$f(0) = 1, f(-1) = -1$$

By the IVT, $\exists x_0 \in [-1, 0], f(x_0) = 0$

Suppose there exists $x_1 \neq x_0$ such that $f(x_1) = 0$. Since f is differentiable, we can apply the MVT to see that there exists $z \in (x_0, x_1)$.

$$f'(z) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = 0$$

But $f'(x) = x^4 + 1 > 0$, so $f'(z) > 0$, but $f'(z) = 0$, a contradiction, and therefore there is at most one root.

Cauchy Mean Value Theorem

Suppose $f, g : [a, b] \to \mathbb{R}$ are continuous and differentiable on (a, b) and $\forall x \in (a, b), g'(x) \neq 0$. Then, $\exists x_0 \in (a, b)$:

$$\frac{f'(x_0)}{g'(x_0)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Proof

Let

$$h(x) = f(x) - g(x)\frac{f(b) - f(a)}{g(b) - g(a)}$$

$$g(b) - g(a) \neq 0$$

because otherwise, there exists $x_1 \in (a, b)$ such that $g'(x_1) = 0$ by Rolle's theorem.

Note that $h(a) = h(b)$, which can be verified with algebra.

By Rolle's theorem, $\exists x_0 \in (a, b)$ such that $h'(x_0) = 0$, and therefore $f'(x_0) - g'(x_0)\frac{f(b)-f(a)}{g(b)-g(a)} = 0$

And finally:

$$\frac{f'(x_0)}{g'(x_0)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Vague Taylor-series-like statement

Let $f : I \to \mathbb{R}$ be n-times differentiable such that for some point $x_0 \in I$

$$f(x_0) = f'(x_0) = f''(x_0) = \cdots = f^{(n-1)}(x_0) = 0$$

Then for any $x \neq x_0$, $x \in I$, there exists z between x an x_0, such that

$$f(x) = \frac{f^{(n)}(z)}{n!}(x - x_0)^n$$

Proof

Let $g(x) = (x - x_0)^n$. By the CMVT, there exists $x_1 \in (x_0, x)$ such that $\frac{f'(x_1)}{g'(x_1)} = \frac{f(x)-f(x_0)}{g(x)-g(x_0)} = \frac{f(x)}{g(x)}$

By the CMVT applied to f' and g' in (x_0, x_1). Then, there exists $x_2 \in (x_0, x_1)$:

$$\frac{f''(x_2)}{g''(x_2)} = \frac{f'(x_1) - f'(x_0)}{g'(x_1) - g'(x_0)} = \frac{f'(x_1)}{g'(x_1)} = \frac{f(x)}{g(x)}$$

Apply n times to get $x_n \in (x_0, x)$ such that

$$\frac{f^{(n)}(x_n)}{g^{(n)}(x_n)} = \frac{f(x)}{g(x)}$$

$$\frac{f^{(n)}(x_n)}{n!} = \frac{f(x)}{(x - x_0)^n}$$

$$f(x) = \frac{f^{(n)}(x_n)}{n!}(x - x_0)^n$$

$z = x_n$, and we have proven the theorem.