MATH341 - 0201

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Contents

Chapter 1: Review and Preliminaries

Complex Numbers

$$\begin{aligned}
\mathbb{C} &= \{a + bi \mid a, b \in \mathbb{R}\} \\
i^2 &= -1 \\
(a + bi) + (c + di) &= (a + c) + (b + d)i \\
(a + bi)(c + di) &= (ac - bd) + (ad + bc)i \\
a &= \text{The real part of } a + bi &&= \text{Re}(a + bi) \\
b &= \text{The imaginary part of} a + bi &&= \text{Im}(a + bi) \\
a + bi = c + di &\Leftrightarrow a = c \wedge b = d
\end{aligned}$$

Figure 1

Definition: Linear Dependence

If $\vec{v}_1, ..., \vec{v}_n$ are linear dependent:

1. One of $\vec{v}_1, ..., \vec{v}_n$ is a linear combination of the others.
2. $\mathbb{c}_1\vec{v}_1 + \cdots + \mathbb{c}_n\vec{v}_n = \vec{0}$ for some $\mathbb{c}_1, ..., \mathbb{c}_n \in \mathbb{F}$

Field properties of complex numbers

For every $x, y, x \in \mathbb{C}$

1. (Commutativity) $x + y = y + x$, $xy = yx$
2. (Associativity) $(x + y) + z = x + (y + z)$, $(xy)z = x(yz)$
3. (Additive Identity) $x + 0 = x$ where $0 = 0 + 0i$
4. (Additive Inverse) There is $t \in \mathbb{C}$ for which $x + t = 0$. When $x = a + bi$, $t = -a + (-b)i$
5. (Multiplicative Inverse) If $x \neq 0$, there is some $t \in \mathbb{C}$ for which $xt = 1$. t is denoted by x^{-1} or $\frac{1}{x}$
6. (Distributivity) x(y + z) = x y + x z

Properties of complex conjugate and norm

For every $z, w \in \mathbb{C}$

1. $\overline{zw} = \overline{z}\ \overline{w}$
2. $|zw| = |z|\ |w|$
3. $|z|^2 = z\overline{z}$
4. (Triangle Inequality) |z+w| <= |z| + |w|

Example

Find the additive and multiplicative inverse of 3 + 2i.

$$(3+2i) + \underbrace{-3+(-2)i}_{\text{inverse}} = 0$$

$$\frac{3+2i}{3+2i} = \frac{(3+2i)(3-2i)}{(3+2i)(3-2i)} = (3+2i)\underbrace{\frac{3-2i}{13}}_{\text{inverse}} = \frac{13}{13} = 1$$

Theorem

$$\begin{aligned}
&\cos(\theta) + i\sin(\theta) \\
&= 1 + \frac{i\theta}{1!} - \frac{i\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} + \cdots \\
&= 1 + \frac{i\theta}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots \\
&= e^{i\theta}
\end{aligned}$$

therefore $\cos(\theta) + i\sin(\theta) = e^{i\theta}$ via

Theorem

1. $e^{ix}e^{iy} = e^{i(x+y)}$
2. (De Moivre's Forumla)$\left(e^{ix}\right)^n = \left(e^{inx}\right)$

Example

Evaluate $\int e^x \cos(x)dx$.

$$e^x\cos(x) = \mathrm{Re}\left(e^{(1+i)x}\right)$$

$$\begin{aligned}
\int e^x\cos(x)dx = \mathrm{Re}\left(\int e^{(1+i)x}dx\right) &\left(\text{via} \int f(x) + g(x)dx = \int f(x)dx + \int g(x)dx\right) \\
= \mathrm{Re}\left(\frac{e^{(1+i)x}}{1+i} + \mathbb{c}\right) \\
= \mathrm{Re}\left(\frac{e^{(1+i)x}}{2}(1-i) + \mathbb{c}\right) \\
= \mathrm{Re}\left(\frac{1}{2}(1-i)(e^x)(\cos(x) + i\sin(x)) + \mathbb{c}\right) \\
= \frac{1}{2}e^x(\cos(x) + \sin(x)) + \mathbb{c}_1
\end{aligned}$$

Vector Spaces, Subspaces, Bases

$$\mathbb{F} = \mathbb{R} \text{ or } \mathbb{C}; \vec{u}, \vec{v} \in V, c \in \mathbb{F}$$
$$\vec{u} + \vec{v} \in V$$
$$c\vec{v} \in V$$

If $\mathbb{F} = \mathbb{R}$, we say that V is a real vector space. If $\mathbb{F} = \mathbb{C}$, we say V is a complex vector space.

Examples of vector spaces

$\mathbb{R}^n$ is a real vector space. $\mathbb{C}^n$ is a complex vector space.

$M_{m \times n}(\mathbb{R})$ is a real vector space. ($+$ is a matrix addition and $\cdot$ is entry wise).

$M_n(\mathbb{R}) = M_{nxn}(\mathbb{R})$ is a real vector space.

$P_n(\mathbb{R}) = \{a_0 + a_1 t + \cdots + a_n t^n \mid a_0, ..., a_n \in \mathbb{R}\}$ is a real vector space.

$C(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous}\}$ is a real vector space.

$C^m(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} : f \text{ is } C^m\}$ is a real vector space.

Subspace Criterion

1. $\mathbb{R}^n$ is a subspace of $\mathbb{C}^n$ as a real vector space (not closed with a complex scalar).
2. $C^1(\mathbb{R})$ is a subspace of $C(\mathbb{R})$

   $C^1(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is } C^1\}$

   $C(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous}\}$

   $0_{C(\mathbb{R})} =$ the constant function 0

   If $f, g \in C^1(\mathbb{R})$ then $(f+g)' = f' + g'$ is continuous. Also, $(cf)' = cf'$ is continuous $\forall c \in \mathbb{R}$

Example: Find a basis for $\mathbb{C}$ once as a real and once as a complex vector space.

1. $\mathbb{C}$ as a real vector space.

We claim $\{1, i\}$ is a basis for $\mathbb{C}$. Now prove linear independence and spanning.

Linear independence

Assume $\mathbb{c}_1 1 + \mathbb{c}_2 i = 0$ for some $\mathbb{c}_1, \mathbb{c}_2 \in \mathbb{R}$.

Then, based on the definition of equals for complex numbers, $\mathbb{c}_1 = \mathbb{c}_2 = 0, \therefore 1, i$ are linearly independent.

Spanning

Let $x \in \mathbb{C}$. By definition, $x = a + bi$ for some $a, b \in \mathbb{R} \Rightarrow x$ is a linear combination of $1, i$. Thus, $1, i$ is a basis for $\mathbb{C}$.

Therefore, $\{1, i\}$ is basis for $\mathbb{C}$, and $\dim_{\mathbb{R}} \mathbb{C} = 2$ (the dimension of $\mathbb{C}$ as real vector space is 2).

2. {1} is a basis for $\mathbb{C}$ as a complex vector space.

Linear independence

$\mathbb{c}_1 1 = 0 \Rightarrow \mathbb{c}_1 = 0 \Rightarrow \{1\}$ is linear independent.

Spanning
Let $x \in \mathbb{C}$. Then $x = x1 \Rightarrow \{1\}$ is spanning.

Example: Span

$\vec{v}_1, ..., \vec{v}_n \in V$

$\text{Span}\{\vec{v}_1, ..., \vec{v}_n\} = \left\{\sum_{j=1}^{n} \mathbb{c}_j \vec{v}_j \mid \mathbb{c}_1, ..., \mathbb{c}_n \in \mathbb{F}\right\}$ is a subspace of F

Linear Transformation and Matrices

These statements are equivalent:

1. $T(\vec{u} + c\vec{v}) = T(\vec{u}) + cT(\vec{v})$
2. $T(a\vec{u} + b\vec{v}) + aT(\vec{u}) + bT(\vec{v})$
3. T is linear

Example 1.18

a. $T(\vec{u}) = A\vec{u}; A \in M_{m\times n}(\mathbb{F})$ is fixed, $T : \mathbb{F}^n \to \mathbb{F}^m$
b. S: $C(\mathbb{R}) \to C(\mathbb{R}); \underbrace{S(f)}_{\text{function from } \mathbb{R}\to\mathbb{R}} (x) = \int_0^x f(t)dt$ is linear

$$\text{Let } f, g \in C(\mathbb{R}), c \in \mathbb{R}$$

$$S(f + cg)(x) = \int_0^x (f + cg)(t)dt = \int_0^x f(t) + cg(t)dt$$

$$= \int_0^x f(t) + c\int_0^x g(t)dt = S(f)(x) + cS(g)(x)$$

$$\Rightarrow S(f + cg) = S(f) + cS(g)$$

$$\Rightarrow S \text{ is linear } \square$$

c. $L : C^1(\mathbb{R}) \to C(\mathbb{R}); L(f)(x) = f'(x)$.
d. $U : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}), U(f)(x) = f''(x) + (2x + 1)f'(x) - e^x f(x)$

Let $f, g \in C^\infty(\mathbb{R}), c \in \mathbb{R}$

$$\begin{aligned} U(f + cg)(x) &= (f + cg)''(x) + (2x + 1)(f + cg)'(x) - e^x(f + cg)(x) \\ &= \underline{f''(x)} + cg''(x) + \underline{(2x + 1)f'(x)} + c(2x + 1)g'(x) - \underline{e^x f(x)} - ce^x g(x) \\ &= \underline{f''(x) + (2x + 1)f'(x) - e^x f(x)} + cg''(x) + c(2x + 1)g'(x) - ce^x g(x) \\ &= U(f)(x) + cU(g)(x) \\ &\Rightarrow U(f + cg) = U(f) + cU(g) \Rightarrow \text{U is linear } \square \end{aligned}$$

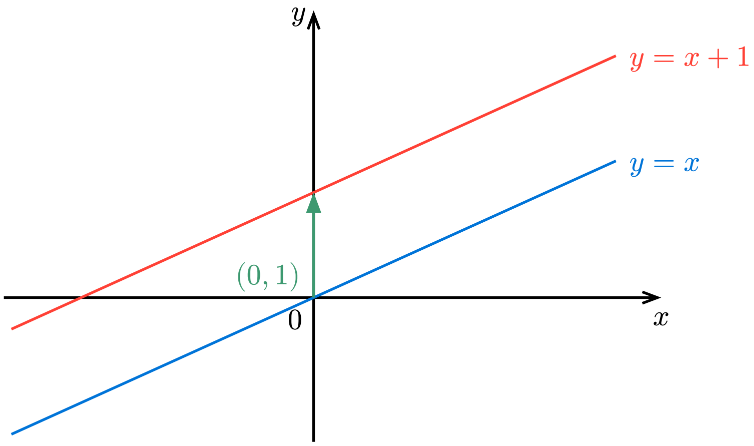
To solve $y'' + (2x + 1)y' - e^x y = 0$ we need to find $\text{Ker } U$.

Definition of tr, trace

Define $\text{tr} : M_{n(\mathbb{F})} \to \mathbb{F}$ called trace by $\text{tr}(A)$ is the sum of it's diagonal entries. Then tr is linear.

$$\text{tr}\left(\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}\right) = \sum_{j=1}^{n} a_{jj}$$

Flatness Theorem



Example

$$T : \mathbb{R}^2 \to \mathbb{R}$$
$$T(x, y) = y - x \text{ is linear}$$
$$\text{The line } y = x \text{ is Ker } T = T^{-1}(\{0\}) = \{(x, y) \in \mathbb{R}^2 \mid T(x, y) = 0\}$$
$$\text{The line } y = x + 1 \text{ is } T^{-1}(\{1\}) = \{(x, y) \in \mathbb{R}^2 \mid T(x, y) = 1\}$$

Theorem

Let $T : V \to W$ be a linear transformation between vector spaces, and let $\vec{w} \in W$. Then either the inverse image $T^{-1}(\{\vec{w}\})$ is empty or $T^{-1} = \vec{v} + \text{Ker } T = \{\vec{v} + \vec{u} \mid \vec{u} \in \text{Ker } T\}$ for every $\vec{v} \in T^{-1}(\{\vec{w}\})$

Proof

$$\text{If } T^{-1}(\{\vec{w}\}) = \varnothing, \text{ we are done!}$$
$$\text{Assume } \vec{v} \in T^{-1}(\{\vec{w}\}).$$
$$\text{Let } \vec{x} \in T^{-1}(\{\vec{w}\})$$
$$\text{Thus, } T(\vec{x}) \in \{\vec{w}\} \Rightarrow T(\vec{x}) = \vec{w}$$
$$\text{Since } \vec{v} \in T^{-1}(\{\vec{w}\}), T(\vec{v}) \in T(\vec{v}) \in \{\vec{w}\} \Rightarrow T(\vec{v}) = \vec{w}.$$
$$T(\vec{x} - \vec{v}) = T(\vec{x}) - T(\vec{v}) = \vec{w} - \vec{w} = 0 \Rightarrow$$
$$\vec{x} - \vec{v} \in \text{Ker } T \Rightarrow \vec{x} = \vec{v} + (\vec{x} - \vec{v}) \Rightarrow \vec{x} \in \vec{v} + \text{Ker } T$$
$$\therefore T^{-1}(\{\vec{w}\})$$
$$\text{Suppose } \vec{x} \in \vec{v} + \text{Ker } T \Rightarrow \vec{x} = \vec{v} + \vec{u} \text{ for some } \vec{u} \in \text{Ker } T$$
$$\Rightarrow T(\vec{x}) = T(\vec{v}) + T(\vec{u}) \text{ by linearity of } T$$
$$= \vec{w} + \vec{0} \because \vec{v} \in T^{-1}(\{\vec{w}\}) \text{ and } \vec{u} \in \text{Ker } T$$
$$\Rightarrow \vec{x} \in T^{-1}(\{\vec{w}\})$$
$$\text{Thus, } \vec{v} + \text{Ker } T \subset T^{-1}(\{\vec{w}\}) \therefore \vec{v} + \text{Ker } T = T^{-1}(\{\vec{w}\})$$

Isomorphism

A linear transformation that is bijective.

Example

$T : P_n \to \mathbb{F}^{n+1}, T(a_0 + a_1 t + \cdots + a_n t^n) = (a_0, ..., a_n)$ is an isomorphism.

Because it preserves addition and scalar multiplication, its "basically" the same structure.

Chapter 2: Diagonalization

Coordinate vector

Let $B = \left(\vec{b_1}, ..., \vec{b_n}\right)$ be an ordered basis for a vector space V. The coordinate vector of a vector $\vec{v} \in V$ relative to B is a column vector $\begin{pmatrix} \mathbb{c}_1 \\ \vdots \\ \mathbb{c}_n \end{pmatrix}$ such that $\vec{v} = \mathbb{c}_1 \vec{v}_1 + \cdots + \mathbb{c}_n \vec{v}_n$. We write $[\vec{v}]_N = \begin{pmatrix} \mathbb{c}_1 \\ \vdots \\ \mathbb{c}_n \end{pmatrix}$

Example

Find $[2t+1]_B$ where $B = (1, 1+t)$ is a basis for P_1.

Solution: We need to write $2t + 1 = \mathbb{c}_1 1 + \mathbb{c}_2 (1 + t) = (\mathbb{c}_1 + \mathbb{c}_2) + \mathbb{c}_2 t$.

$$\begin{cases} \mathbb{c}_1 + \mathbb{c}_2 = 1 \\ \mathbb{c}_2 = 2 \end{cases} \to \begin{cases} \mathbb{c}_1 = -1 \\ \mathbb{c}_2 = 2 \end{cases}$$

$$[2t+1]_B = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

Coordinate vectors are isomorphisms

$B = \left(\vec{b_1}, ..., \vec{b_n}\right)$ be an ordered basis for a vector space V. Then $[.]_B : V \to \mathbb{F}^n$ is an isomorphism.

Proof

Linearity:

$$\text{Let } \vec{v}, \vec{w} \in V, c \in \mathbb{F}$$

$$\text{Since } B \text{ is a basis } \vec{v} = \sum_{j=1}^{n} \mathbb{c}_j \vec{b_j} \text{ and } \vec{w} = \sum_{j=1}^{n} \mathbb{d}_j \vec{b_j} \Rightarrow \vec{v} + c\vec{w} = \sum_{j=1}^{n} (\mathbb{c}_j + c\mathbb{d}_j) \vec{b_j}$$

$$\text{Thus, } [\vec{v} + c\vec{w}] = \begin{pmatrix} \mathbb{c}_1 + c\mathbb{d}_1 \\ \vdots \\ \mathbb{c}_n + c\mathbb{d}_n \end{pmatrix} = \begin{pmatrix} \mathbb{c}_1 \\ \vdots \\ \mathbb{c}_n \end{pmatrix} + c \begin{pmatrix} \mathbb{d}_1 \\ \vdots \\ \mathbb{d}_n \end{pmatrix} = [\vec{v}]_B + c[\vec{w}]_B$$

One-to-one:

$$\text{Suspose } [\vec{v}]_B = [\vec{w}]_B = \begin{pmatrix} \mathbb{c}_1 \\ \vdots \\ \mathbb{c}_n \end{pmatrix}$$

$$\text{By definition of } []_B, \vec{v} = \sum_{j=1}^{n} \mathbb{c}_j \vec{b}_j \text{ and } \vec{w} = \sum_{j=1}^{n} \mathbb{c}_j \vec{b}_j$$

$$\therefore \vec{v} = \vec{w}.$$

Onto:

$$\text{Let } \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{F}^n$$

$$\text{Then, } \left[\sum_{j=1}^{n} a_j \vec{b}_j\right]_B = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

Change of Coordinates Theorem

Let V, W be vector spaces over $\mathbb{F}$ with ordered bases $A = (\vec{a_1}, ..., \vec{a_n})$ and $B = \left(\vec{b_1}, ..., \vec{b_n}\right)$, respectively. Let $T : V \to W$ be a linear transformation

Then, there is a unique matrix $\underbrace{A}_{M_{m\times n}(\mathbb{F})}$ such that $[T(\vec{v})]_B = A[\vec{v}]_A$ for all $\vec{v} \in V$.

$$\text{Furthermore, } A = \left([T(\vec{a_1})]_B \ [T(\vec{a_2})]_B \ \cdots \ [T(\vec{a_n})]_B\right)$$

Notation: This matrix is denoted by $[T]_{BA}$, $[T(\vec{v})]_B = [T]_{BA}[\vec{v}]_A$

Proof

$$\text{Let } v \in V. \text{ Suppose } \vec{v} = \sum_{j=1}^{n} \mathbb{c}_j \vec{a}_j.$$

$$[T(\vec{v})]_B \underbrace{=}_{T \text{ is linear}} \left[\sum_{j=1}^{n} \mathbb{c}_j T(\vec{a}_j)\right]_B \underbrace{=}_{[\,]_B \text{ is linear}} \sum \mathbb{c}_j [T(\vec{a}_j)]_B$$

$$= \left([T(\vec{a_1})]_B \ [T(\vec{a_2})]_B \ \cdots \ [T(\vec{a_n})]_B\right) \begin{pmatrix} \mathbb{c}_1 \\ \vdots \\ \mathbb{c}_n \end{pmatrix}$$

Uniqueness

$$\text{Suppose } A[\vec{v}]_A = B[\vec{v}]_A \forall v \in V \text{ and } A, B \in M_{m\times n}(\mathbb{F})$$

$$\text{Replace } \vec{v} \text{ by } \vec{a}_j \Rightarrow A[\vec{a}_j]_A = B[\vec{a}_j]_A \Rightarrow A\vec{e}_j = B\vec{e}_j$$

$$\therefore \text{The } j\text{th column of } A = \text{The } j\text{th column of } B \ \square$$

Example

Consider the linear transformation $T : \mathbb{F}^2 \to P_1$ given by $T(a, b) = a + b + (a - b)t$. Write down the matrix of T relative to:

a. $A = (\vec{e}_1, \vec{e}_2)$ for $\mathbb{F}^2$ and $B = (1, t)$ for P_1.

Solution:

$$[T]_{BA} = \left([T(1,0)]_B \ \ [T(0,1)]_B\right) = \left([1+t]_B \ \ [1-t]_B\right) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

b. $A = ((1,1),(0,1))$ for $\mathbb{F}^2$ and $B = (t, 1-t)$ for P_1.

Solution:

$$[T]_{BA} = \left([T(1,1)]_B \ \ [T(0,1)]_B\right) = \left([2]_B \ \ [1-t]_B\right) = \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix}$$

Conversion Theorem

Suppose U, V, W are vector spaces over $\mathbb{F}$ with ordered bases A, B, C, respectively. Let $S : U \to V$ and $T : V \to W$ be linear transformations. Then, $T \circ S$ is linear. Furthermore, $[T \circ S]_{CA} = [T]_{CB}[S]_{BA}$.

Change of Coordinates Matrix

Consider $I : V \to V$ defined by $I(\vec{x}) \to \vec{x}$. Let A, B be ordered bases for V.

$$[I(\vec{v})]_B = [I]_{BA}[\vec{v}]_A \forall \vec{v} \in V$$

$$[\vec{v}]_B = \underbrace{[I]_{BA}}_{\text{The change of coordinate matrix from } A \to B} [\vec{v}]_A$$

Inverse of Change of Coordinates Matrix

$$[I]_{BA} = [I]_{AB}^{-1}$$

Proof

$$[I]_{AB}[I]_{BA} = [I \circ I]_{AA} = [I]_{AA}$$

$$\text{Suppose } A = (\vec{a_1}, ..., \vec{a_n}). \text{ Then } [I]_{AA} = \left([I(\vec{a_1})]_A \ \cdots \ [I(\vec{a_n})]_A\right) = \left([\vec{a_1}]_A \ \cdots \ [\vec{a_n}]_A\right) = (\vec{e_1} \ \cdots \ \vec{e_n}) = I$$

$$\text{by the IVT, } [I]_{BA} = [I]_{AB}^{-1}$$

Example

Write $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ in the ordered basis $B = \left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \end{pmatrix}\right)$ using a change of coordinates matrix.

Solution:

$$\text{Let } S = (\vec{e_1}, \vec{e_2}). \text{ We know } \left[\begin{pmatrix} 2 \\ 3 \end{pmatrix}\right]_S = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

$$\text{We need to find } [I]_{BS} = [I]_{SB}^{-1} = \left([I(1,2)]_S \ \ [I(3,5)]_S\right)^{-1} = \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix}^{-1}$$

$$\left[\begin{pmatrix} 2 \\ 3 \end{pmatrix}\right]_B = \underbrace{\begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix}^{-1}}_{[I]_{BS}} \underbrace{\begin{pmatrix} 2 \\ 3 \end{pmatrix}}_{\left[\binom{2}{3}\right]_S} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Example

Find the change of coordinate matrix from the ordered basis $A = (1, 1 + t)$ to the ordered basis $B = (1 + 2t, 1 - 2t)$ of P_1.

Solution: We are looking for $[I]_{BA}$

$$\text{Let } S = (1, t)$$

$$[I]_{BA} = [I]_{BS}[I]_{SA} = [I]_{SB}^{-1}\left([1]_S \;\; [1+t]\right)$$

$$= \left([1+2t]_S \;\; [1-2t]_S\right)^{-1}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 1\\ 2 & -2\end{pmatrix}^{-1}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} = \begin{pmatrix}\frac{1}{2} & \frac{3}{4}\\ \frac{1}{2} & \frac{1}{4}\end{pmatrix}$$

Similarity

We say matrices A, B are similar if $A = PBP^{-1}$ for some invertible matrix P.

Similarity from Change of Coordinates

Suppose A and B are ordered bases of a vector space V and $T : V \to V$ is a linear transformation.

$$[T]_{BB} = [I \circ T \circ I]_{BB} = [I]_{BA}[T]_{AA}[I]_{AB} = [I]_{BA}[T]_{AA}[I]_{BA}^{-1}$$

$$[T]_{BB} = [I]_{BA}[T]_{AA}[I]_{BA}^{-1}$$

Eigenvectors and Diagonalization

Example

Evaluate $A^{100}\vec{v}$, $A = \begin{pmatrix}2 & 3\\ -4 & 1\end{pmatrix}, \vec{v} = \begin{pmatrix}1\\ -1\end{pmatrix}$

$$A\vec{v} = \begin{pmatrix}2 & 3\\ -4 & 1\end{pmatrix}\begin{pmatrix}1\\ -1\end{pmatrix} = \begin{pmatrix}5\\ -5\end{pmatrix} = 5\vec{v}$$

$$A^2\vec{v} = A(5\vec{v}) = 5A\vec{v} = 5^2\vec{v}$$

$$\vdots$$

$$A^{100}\vec{v} = 5^{100}\vec{v}$$

$\vec{v}$ is an eigenvector, and 5 is its eigenvalue.

Eigenvalues, Eigenvectors and Eigenpairs

Suppose $T : V \to V$ is a linear transformation and $\vec{v} \in V$ is a nonzero vector such that $\exists \lambda \in \mathbb{F}, T(\vec{v}) = \lambda\vec{v}$. We say λ is an eigenvalue, $\vec{v}$ is an eigenvector, and $(\lambda, \vec{v})$ is an eigenpair for T. The same notion is also defined for square matrices.

Eigenvalues from the characteristic polynomial theorem

λ is an eigenvalue for A iff $\det(A - \lambda I) = 0$.

Characteristic Polynomial

The polynomial $\det(A - zI)$ is called the characteristic polynomial of A.

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ & \vdots & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

$$A - \lambda I = \begin{pmatrix} a_{11} - \lambda & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} - \lambda \end{pmatrix} = \text{polynomial } P(\lambda)$$

Example finding Eigenvalues

Find the eigenvalues of $\left(\begin{smallmatrix} 1 & 1 \\ -1 & 1 \end{smallmatrix}\right)$ once as an element of $M_2(\mathbb{R})$ and once as an element of $M_2(\mathbb{C})$.

Over $\mathbb{R}$

$$\det(A - zI) = \det\begin{pmatrix} 1-z & 1 \\ -1 & 1-z \end{pmatrix} = (1-z)^2 + 1 = 0$$

$$(1-z)^2 = -1, \text{which has no solution over } \mathbb{R}$$

Over $\mathbb{C}$

$$(1-z)^2 = -1$$
$$1 - z = \pm i$$
$$z = 1 \pm i \qquad \square$$

Why diagonal matrices are nice

$$\begin{pmatrix} a & & 0 \\ & b & \\ 0 & & c \end{pmatrix} \begin{pmatrix} x & & 0 \\ & y & \\ 0 & & z \end{pmatrix} = \begin{pmatrix} ax & & 0 \\ & by & \\ 0 & & cz \end{pmatrix}$$

So easy 😌.

Diagonalizable Matrix Definition

A square matrix A is said to be diagonalizable if $\exists$ a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$. In other words, A and D are similar.

diag definition

$$\operatorname{diag}(\lambda_1, ..., \lambda_n) = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

Find the Diagonalization of a Matrix

A matrix $A \in M_{n(\mathbb{F})}$ is diagonalizable iff there is a basis for $\mathbb{F}^n$ consisting of eigenvectors of A.

Furthemore, if $(\lambda_1, \vec{v}_1), ..., (\lambda_n, \vec{v}_n)$ are eigenpairs of A whose eigenvectors form a basis for $\mathbb{F}^n$, then $A = PDP^{-1}$ where $P = (\vec{v}_1 \ \cdots \ \vec{v}_n)$ and $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$.

Proof

$$(\Rightarrow) \text{ Suppose } A = PDP^{-1}, D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}, P = (\vec{v}_1 \ \cdots \ \vec{v}_n)$$

$$\Rightarrow AP = PD \Rightarrow A(\vec{v}_1 \ \cdots \ \vec{v}_n) = (\vec{v}_1 \ \cdots \ \vec{v}_n) \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

The jth column: $Av_j = \lambda_j \vec{v}_j \Rightarrow (\lambda_j, \vec{v}_j)$ is an eigenpair. (By the definition of matrix-matrix multiplication)

Since P is invertible, its columns $\vec{v}_1, ..., \vec{v}_n$ form a basis for $\mathbb{F}^n$.

$(\Longleftarrow)$ Suppose $(\lambda_1, \vec{v}_1), ..., (\lambda_n, \vec{v}_n)$ are eigenpairs for A and $\vec{v}_1, ..., \vec{v}_n$ form a basis for $\mathbb{F}^n$

Define $T : \mathbb{F}^n \to \mathbb{F}^n, T(\vec{x}) = A\vec{x}$. Let $S = (\vec{e_1}, ..., \vec{e_n})$ and $B = (\vec{v}_1, ..., \vec{v}_n)$

$$[T]_{SS} = \left([T(\vec{e_1})]_S \ \cdots \ [T(\vec{e_n})]_S\right) = (\text{first column of } A \ \cdots \ n\text{th column of } A) = A$$

$$[T]_{BB} = \left([T(\vec{v}_1)]_B, \cdots, [T(\vec{v}_n)]_B\right) = \left([\lambda_1 \vec{v}_1]_{BB} \ \cdots \ [\lambda_n \vec{v}_n]_B\right) = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

$$A = [T]_{\mathbb{S}} = [I]_{SB}[T]_{\mathbb{B}}[I]_{SB} = (\vec{v}_1 \ \cdots \ \vec{v}_n) \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} (\vec{v}_1 \ \cdots \ \vec{v}_n)^{-1}$$

Example

Diagonalize $A = \begin{pmatrix} 2 & -3 \\ -4 & 1 \end{pmatrix}$. Use that to find $A^n \forall n \in \mathbb{Z}^+$.

Solution

$$\det(A - zI) = \begin{pmatrix} 2-z & -3 \\ -4 & 1-z \end{pmatrix} = (2-z)(1-z) - 12 = (z-5)(z+2) \Rightarrow \text{eigenvalues are } \{5, -2\}$$

$$\underline{\text{For } z = 5:}$$

$$A - 5I = \begin{pmatrix} -3 & -3 \\ -4 & -4 \end{pmatrix} \Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ is an eigenvector}$$

$$\underline{\text{For } z = 3:}$$

$$A - 5I = \begin{pmatrix} 4 & -3 \\ -4 & 3 \end{pmatrix} \Rightarrow \vec{v}_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \text{ is an eigenvector}$$

$$\left(5, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right), \left(-2, \begin{pmatrix} 3 \\ 4 \end{pmatrix}\right) \text{ are eigenpairs. } \begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ and} \begin{pmatrix} 3 \\ 4 \end{pmatrix} \text{ are not scalar multiples} \therefore \text{linear independent}$$

$$\dim \mathbb{F}^2 = 2 \therefore \vec{v}_1, \vec{v}_2 \text{ form a basis for } \mathbb{F}^2$$

$$\text{By a theorem, } A = PDP^{-1} = \begin{pmatrix} 1 & 3 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ -1 & 4 \end{pmatrix}^{-1}$$

$$A = PDP^{-1}$$

$$A^2 = PDP^{-1}PDP^{-1} = PDDP^{-1} = PD^2P^{-1}$$

$$A^3 = PDP^{-1}PDP^{-1}PDP^{-1} = PDDDP^{-1} = PD^3P^{-1}$$

$$\vdots$$

$$A^n = PD^nP^{-1} = \begin{pmatrix} 1 & 3 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} 5^n & 0 \\ 0 & (-2)^n \end{pmatrix} \begin{pmatrix} \frac{4}{7} & -\frac{3}{7} \\ \frac{1}{7} & \frac{1}{7} \end{pmatrix}$$

Example

Show $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ is not diagonalizable.

Solution

$$\det(A - zI) = \det \begin{pmatrix} 1-z & 2 \\ 0 & 1-z \end{pmatrix} = (1-z)^2 \therefore \text{eigenvalues are } 1, 1$$

$$\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow y = 0. \text{ then eigenvectors } \begin{pmatrix} x \\ 0 \end{pmatrix}, x \neq 0, x \in \mathbb{F}$$

$$\therefore \text{no basis of } \mathbb{F}^2 \text{ consisting of eigenvalues}$$

Similar Matrices and Characteristic Polynomial Theorem

Every two similar matrices have the same characteristic polynomial.

Proof

$$\text{Suppose } A, B \text{ are similar. By definition, } \exists P, A = PBP^{-1}.$$

$$\det(A - zI) = \det(PBP^{-1} - zI) = \det(P(B - zI)P^{-1})$$

$$= \det(P)\det(B - zI)\det(P^{-1}) = \det(P)\det(P^{-1})\det(B - zI) = \det(PP^{-1})\det(B - zI)$$

$$= \det(I)\det(B - zI) = \det(B - zI)$$

Distinct Eigenvalues lead to linearly independent Eigenvectors

Eigenvectors corresponding to distinct eigenvalues are linearly independent. Furthermore, if an $n \times n$ matrix has n distinct eigenvalues, then it is diagonalizable.

Proof

Suppose $(\lambda_1, \vec{v}_1), ..., (\lambda_m, \overrightarrow{v_m})$ are eigenpairs corresponding to a matrix A (or a linear transformation T) with $\lambda_1, ..., \lambda_m$ distinct. We will prove $\vec{v}_1, ..., \overrightarrow{v_m}$ are linearly independent by induction on m.

Base case: $m = 1$:

Since $\vec{v}_1 \neq \vec{0}$, it is linearly independent.

Inductive Hypothesis: Assume $\vec{v}_1, ..., \overrightarrow{v_{m-1}}$ are linearly independent.

Inductive Step: Prove $\vec{v}_1, ..., \overrightarrow{v_m}$ are linearly independent.

$$\text{Assume } \mathbb{c}_1\vec{v}_1 + \cdots + \mathbb{c}_m\overrightarrow{v_m} = \vec{0} \text{ for some } \mathbb{c}_1, ..., \mathbb{c}_m \in \mathbb{F}.$$
$$\mathbb{c}_1 A\vec{v}_1 + \cdots + \mathbb{c}_{m-1}A\overrightarrow{v_{m-1}} + \mathbb{c}_m A\overrightarrow{v_m} = A\vec{0} = \vec{0} \qquad (*)$$
$$\mathbb{c}_1\lambda_1\vec{v}_1 + \cdots + \mathbb{c}_{m-1}\lambda_{m-1}\overrightarrow{v_{m-1}} + \mathbb{c}_m\lambda_m\overrightarrow{v_m} = \vec{0} \qquad (**)$$
$$\mathbb{c}_1\lambda_1\vec{v}_1 + \cdots + \mathbb{c}_{m-1}\lambda_{m-1}\overrightarrow{v_{m-1}} + \mathbb{c}_m\lambda_m\overrightarrow{v_m} = \vec{0}$$
$$\lambda_m(*) - (**) \text{ yields } \mathbb{c}_1(\lambda_m - \lambda_1)\vec{v}_1 + \cdots + \mathbb{c}_{m-1}(\lambda_m - \lambda_{m-1})\overrightarrow{v_{m-1}} = \vec{0}$$
$$\text{By the I.H. } \mathbb{c}_1(\lambda_m - \lambda_1) = \cdots = \mathbb{c}_{m-1}(\lambda_m - \lambda_{m-1}) = 0$$
$$\text{Since } \lambda_m \neq \lambda_1, ..., \lambda_m \neq \lambda_{m-1}, \mathbb{c}_1 = \cdots = c_{m-1} = 0 \underset{(*)}{\Rightarrow} \mathbb{c}_m = 0 \qquad \square.$$

Exponentiating Matrix

$A \in M_n(\mathbb{F})$

We want to define e^A.

By the Taylor series:

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} = I + \frac{A}{1!} + \frac{A^2}{2!} + \cdots$$

Then if $A = P\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}P^{-1}$ then $e^A = P\begin{pmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{pmatrix}P^{-1}$.

Proof for Diagonalizable Matrices

$$\text{Suppose } D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

$$e^D = \sum_{k=0}^{\infty} \frac{D^k}{k!} = \sum_{k=0}^{\infty} \frac{\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}^k}{k!} = \begin{pmatrix} \sum_{k=0}^{\infty} \frac{\lambda_1^k}{k!} & & 0 \\ & \ddots & \\ 0 & & \sum_{k=0}^{\infty} \frac{\lambda_n^k}{k!} \end{pmatrix} = \begin{pmatrix} e_1^{\lambda} & & 0 \\ & \ddots & \\ 0 & & e_n^{\lambda} \end{pmatrix}$$

$$\text{Suppose } A = PDP^{-1}$$

$$e^A = \sum_{k=0}^{\infty} \frac{\left(PDP^{-1}\right)^k}{k!} = \sum_{k=0}^{\infty} \frac{PD^kP^{-1}}{k!} = P\left(\sum_{k=0}^{\infty} \frac{D^k}{k!}\right)P^{-1} = P\begin{pmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{pmatrix}P^{-1}$$

$$\therefore \text{If } A = P\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}P^{-1} \text{ then } e^A = P\begin{pmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{pmatrix}P^{-1}$$

Why we may want this:

$$y(t) = e^t$$

$$\begin{cases} y' = y \\ y(0) = 1 \end{cases} \qquad \begin{cases} \frac{d(e^{tA})}{dt} = Ae^{tA} \\ e^{tA} \;|_{t=0} = I \end{cases}$$

Example

Evaluate e^A where $A = \left(\begin{smallmatrix} 2 & -3 \\ -4 & 1 \end{smallmatrix}\right)$.

Solution

We previously did this, therefore:

$$A = \begin{pmatrix} 3 & -1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} -2 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ 4 & 1 \end{pmatrix}^{-1}$$

$$e^A = \begin{pmatrix} 3 & -1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} e^{-2} & 0 \\ 0 & e^5 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ 4 & 1 \end{pmatrix}^{-1}$$

Fundamental Theorem of Algebra

Every polynomial of degree n with complex coefficients can be completely factored into linear terms.

In other words, if $p(t) = a_n t^n + \cdots + a_1 t + a_0$ with $a_j \in \mathbb{C}$ and $a_n \neq 0$, then $p(t) = a_n(t - \mathbb{c}_1)\cdots(t - \mathbb{c}_n)$ for some $\mathbb{c}_j \in \mathbb{C}$.

Upper and Lower Triangular

A square matrix is called upper triangular if every (j, k) entry with $j > k$ is zero.

A square matrix is called lower triangular if every (j, k) entry with $j < k$ is zero.

Chapter 3: Jordan Form

Block Multiplication of Matrices

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{pmatrix}$$

For every two block matrices, as long as all matrix operations are valid. A similar result holds for all other block matrices.

Theorem: Schur's Theorem

Every square matrix in $M_{n(\mathbb{C})}$ is similar to an upper triangular matrix.

Proof

Induction on n:

Let $A \in M_n(\mathbb{C})$.

Base case $n = 1$: A is already upper triangular.

Inductive Hypothesis: Assume the result holds for matrices in $M_{n-1}(\mathbb{C}), 1 \leq n - 1 \leq n$

Inductive Step:

Let $\lambda_1 \in \mathbb{C}$ be a root of the characteristic polynomial of A. Let $\vec{v}_1 \in \mathbb{C}^n$ be an eigenvector corresponding to λ_1. Let $B = (\vec{v}_1, ..., \vec{v}_n)$ be an ordered basis for $\mathbb{C}^n$. Define $T : \mathbb{C}^n \to \mathbb{C}^n, T(\vec{x}) = A\vec{x}$. Let $S = (\vec{e_1}, ..., \vec{e_n})$

$$[T]_{SS} = A \text{ by a theorem}$$

$$[T]_{BB} = \left([T(\vec{v}_1)]_B \ \cdots\right) = \left([A\vec{v}_1]_B \ \cdots\right) = ([\lambda_1\vec{v}_1] \ \cdots) = \begin{pmatrix} \lambda_1 & \\ 0 & * \end{pmatrix}$$

$$A \text{ is similar to } \begin{pmatrix} \lambda_1 & \\ 0 & * \end{pmatrix} \text{ by a theorem}$$

$$A = P\begin{pmatrix} \lambda_1 & V \\ 0 & B \end{pmatrix}P^{-1}, B \in M_{n-1}(\mathbb{C}), V \in M_{1\times(n-1)}(\mathbb{C})$$

$$\text{By the inductive hypothesis, } B = QCQ^{-1} \text{ where } C \text{ is upper triangular}$$

$$A = P\begin{pmatrix} \lambda_1 & V \\ 0 & QCQ^{-1} \end{pmatrix}P^{-1}$$

$$A = P\begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix} \underbrace{\begin{pmatrix} \lambda_1 & W \\ 0 & C \end{pmatrix}}_{\text{upper triangular}} \begin{pmatrix} 1 & 0 \\ 0 & Q^{-1} \end{pmatrix}P^{-1}$$

$$\begin{pmatrix} \lambda_1 & W \\ 0 & QC \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & Q^{-1} \end{pmatrix} = \begin{pmatrix} \lambda_1 & WQ^{-1} \\ 0 & QCQ^{-1} \end{pmatrix}$$

$$\text{We need } WQ^{-1} = V \iff W = VQ$$

$$\text{Plug in and get the upper triangular matrix}$$

Example

Write the following matrix as PTP^{-1}, where T is upper triangular.

$$A = \begin{pmatrix} 0 & 0 & -1 \\ 5 & 2 & 3 \\ 2 & 0 & 3 \end{pmatrix}$$

Solution

$$\det(A - zI) = \det\begin{pmatrix} -z & 0 & -1 \\ 5 & 2-z & 3 \\ 2 & 0 & 3-z \end{pmatrix}$$

$$(-1)^{2+2}(2-z)(-z(3-z)+2) = (2-z)(z^2-3z+2) = (2-z)(z-1)(z-2)$$

Find an eigenpair $(1, \vec{v}_1)$. Extend to a basis to obtain $\vec{v}_2, \vec{v}_3$, then follow the proof $\square$

Recall: Existence of Upper Triangular Matrix

If $A \in M_n(\mathbb{C})$, then $a = PTP^{-1}$ for some upper triangular matrix T.

Example:

Give an example of a $M_n(\mathbb{R})$ matrix that is not similar to an upper triangular matrix in $M_n(\mathbb{R})$.

Solution

Rotation matrix:

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Eigenvalues are $\pm i$. If $A = P\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}P^{-1}$ then $\lambda_1, \lambda_2 = \pm i$ since similar matrices have the same characteristic polynomial, thus the same eigenvalues.

Definition: matrix input into a polynomial

Let $A \in M_n(\mathbb{F})$ and $P(A) = a_0 I + a_1 A + \cdots + a_m A.^m$ be a polynomial with $a_j \in \mathbb{F}$.

Cayley-Hamilton theorem matrix

Let $p(z)$ be the characteristic polynomial of a matrix $A \in M_n(\mathbb{F})$. Then $p(A) = 0$. To evaluate P(A), *first* evaluate $\det(A - zI)$, *then* substitute $z = A$.

Proof

By Schur's Theorem, there is an invertible matrix $S \in M_n(\mathbb{C})$ and an upper triangular matrix $T \in M_n(\mathbb{C})$ such that $A = STS^{-1}$. By a theorem, the characteristic polynomial of T is also $p(z)$.

Assume $p(z) = a_0 + \cdots + a_n z^n$.

$$\begin{aligned} P(A) &= a_0 I + a_1 A + \cdots + a_n A^n \\ &= a_0 I + a_1 PTS^{-1} + \cdots + a_n PT^n S^{-1} \\ &= S(p(T))S^{-1} \end{aligned}$$

We need to show $p(T) = 0$.

Base case $n = 1$.

$T = (\lambda_1), p(z) = \lambda_1 - z, P(T) = \lambda_1 I - (\lambda_1) = 0$

Inductive Step:

Now, we will prove the theorem for upper triangular matrices by induction on n.

$$T=\begin{pmatrix}\lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n\end{pmatrix}=\begin{pmatrix}\lambda_1 & * \\ 0 & B\end{pmatrix}, B=\begin{pmatrix}\lambda_2 & & * \\ & \ddots & \\ 0 & & \lambda_n\end{pmatrix}\in M_{n-1}(\mathbb{C})$$

$$\text{Eigenvalues of } T \text{ are } \lambda_1,...,\lambda_n \Rightarrow p(z)=\det(T-zI)=(\lambda_1-z)\cdots(\lambda_n-z)$$

$$q(z)=\det(B-zI)=(\lambda_2-z)\cdots(\lambda_n-z)$$

$$\therefore p(z)=(\lambda_1-z)q(z)P(T)=(\lambda_1 I-T)Q(T)=\begin{pmatrix}0 & & & * \\ & \lambda_1-\lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_1-\lambda_n\end{pmatrix}\begin{pmatrix}q(\lambda_1) & * \\ 0 & q(B)\end{pmatrix}$$

$$=\begin{pmatrix}0 & & & * \\ & \lambda_1-\lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_1-\lambda_n\end{pmatrix}\begin{pmatrix}q(\lambda_1) & * \\ 0 & \underbrace{0}_{\text{By I.H.}}\end{pmatrix}=\underbrace{0}_{\text{first column 0 × second and on rows zero}}$$

□

Jordan Canonical Form

Objective: Write every matrix

$$A=P\begin{pmatrix}B_1 & & 0 \\ & \ddots & \\ 0 & & B_m\end{pmatrix}P^{-1}$$

$$B_j=\begin{pmatrix}\lambda_1 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_j\end{pmatrix}$$

Each B_j is called a Jordan block.

Example

$$\text{Consider } A=\begin{pmatrix}2 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2\end{pmatrix}$$

a. Find all eigenpairs of A. Show A is not diagonalizable.

All eigenvalues are 2 because this matrix is diagonal, and those are along the diagonal.

$\text{Ker }(A-2I)=\text{Span }\{\vec{e_1},\vec{e_2}\}$

Eigenpairs $=\left(2,\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\right),\left(2,\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\right)$

Since $\dim \text{Ker }(A-2I)=2\neq 4$ there is no basis of $\mathbb{F}^4$ of eigenvectors. $\therefore A$ is not diagonalizable.

b. Find a basis for $\text{Ker }\left((A-2I)^2\right)$

$$(A-2I)^2 = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{Ker }\left((A-2I)^2\right) = \text{Span }\{\vec{e_1}, \vec{e_2}, \vec{e_3}\}$$

c. Find a basis for $\text{Ker }((A-2I)^n)\forall n \in \mathbb{Z}^+$

$$(A-2I)^3 = 0 \Rightarrow \text{Ker }(A-2I)^3 = \mathbb{F}^4$$

$$\therefore \text{ a basis for Ker }((A-2I)^n) = \{\vec{e_1}, \vec{e_2}, \vec{e_3}, \vec{e_4}\}\forall n \in \mathbb{Z}^{z\geq 3}$$

d. Find A in an "almost diagonal form".

$$A = P\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}P^{-1} \Rightarrow$$

$$\begin{cases} T(\vec{v}_1) = 2\vec{v}_1 \\ T(\vec{v}_2) = 2\vec{v}_2 \\ T(\vec{v}_3) = \vec{v}_2 + 2\vec{v}_3 \Rightarrow (A-2I)v_3 = v_2 \\ T(\vec{v}_3) = \vec{v}_3 + 2\vec{v}_4 \Rightarrow (A-2I)v_4 = v_4 \end{cases}$$

Start with a vector in $\text{Ker }(A-2I)^3$ that is not in $\text{Ker }(A-2I)^2$. Call that $\vec{v}_4 : \vec{v}_4 = \vec{e_4}$. Set $\vec{v}_3 = (A-2I)\vec{v}_4 = 2\vec{e_3}$

$$\vec{v}_2 = (A-2I)\vec{v}_3 = 2\vec{e_1} - 2\vec{e_2}$$

$$(A-2I)\vec{v}_2 = \vec{0}$$

$$\text{To summarize: } \begin{cases} A\vec{v}_2 = 2\vec{v}_2 \\ A\vec{v}_3 = \vec{v}_2 + 2\vec{v}_3 \\ A\vec{v}_4 = \vec{v}_3 + 2\vec{v}_4 \end{cases}$$

$$\therefore \vec{v}_2 = \begin{pmatrix} 2 \\ -2 \\ 0 \\ 0 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \vec{v}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\text{Choose } \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ Set } \mathcal{B} = (\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4) \text{ a basis for } \mathbb{F}^4$$

$$\text{Consider } T : \mathbb{F}^4 \to \mathbb{F}^4 \text{ by } T(\vec{x}) = A\vec{x}$$
$$[T]_{SS} = A, S = (\vec{e_1}, \vec{e_2}, \vec{e_3}, \vec{e_4})$$
$$[T]_{\mathcal{B}\mathcal{B}} = \left([T(\vec{v}_1)]_{\mathcal{B}} \ [T(\vec{v}_2)]_{\mathcal{B}} \ [T(\vec{v}_3)]_{\mathcal{B}} \ [T(\vec{v}_4)]_{\mathcal{B}}\right)$$
$$= \left([2\vec{v}_1]_B \ [2\vec{v}_2]_B \ [\vec{v}_2 + 2\vec{v}_3]_B \ [\vec{v}_4 + 2\vec{v}_4]_B\right) = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$
$$[T]_{SS} = [I]_{SB}[T]_{BB}[I]_{SB}^{-1} = \underbrace{\begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{P} \underbrace{\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}}_{T} \underbrace{\begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1}}_{P^{-1}} \quad \square$$

Theorem: Steady state kernel space

Let λ be an eigenvalue of an $n \times n$ matrix A. Then, $\exists k \leq n$ such that

$$\operatorname{Ker}(A - \lambda I) \subsetneq \operatorname{Ker}\ (A - \lambda I)^2 \subsetneq \cdots \subsetneq \operatorname{Ker}(A - \lambda I)^k = \operatorname{Ker}(A - \lambda I)^{k+1} = \cdots$$

Generalized Eigenvector

For an eigenvalue λ, $A \in M_n(\mathbb{F})$, every nonzero vector in $\operatorname{Ker}\ (A - \lambda I)^n$ is called a generalized eigenvector. The vector space $\operatorname{Ker}\ (A - \lambda I)^n$ is called the generalized eigenspace associated to λ.

Theorem: linearly independent generalized eigenvectors

Generalized eigenvectors corresponding to distinct eigenvalues are linearly independent.

The dimension of the generalized eigenspace

The dimension of the generalized eigenspace corresponding to an eigenvalue λ is the same as the multiplicity of λ as a root of character polynomials.

Existence of Jordan Canonical Form

Every matrix $A \in M_n(\mathbb{C})$ has a Jordan Decomposition. Furthermore, this matrix in Jordan form is unique up to permutations of Jordan blocks.

In other words:

$$A = PJP^{-1}; J = \begin{pmatrix} B_1 & & 0 \\ & \ddots & \\ 0 & & B_m \end{pmatrix}; B_j = \begin{pmatrix} \lambda_j & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_j \end{pmatrix}$$

Example

Find a matrix in Jordan form that is similar to

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 5 & -1 \\ -4 & 13 & -2 \end{pmatrix}$$

Find P such that $A = PJP^{-1}$ where J is in Jordan form

Solution

$$\det(A - ZI) = -z^3 + 5z^2 - 8z + 4 \underbrace{=}_{\text{rational root \& long division}} (z-1)(-z+2)(z-2) \therefore \text{eigenvalues are 1,2,2}$$

$$\text{Ker } (A - 1I) = \text{Span}\left\{\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}\right\} \qquad 1 = 1, \text{we are done}$$

$$\text{Ker } (A - 2I) = \text{Span}\left\{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\right\} \qquad 1 \neq 2, \text{we must continue}$$

$$\text{Ker } (A - 2I)^2 = \text{Span}\left\{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}\right\}, \qquad 2 = 2, \text{we are done}$$

$$A - 2I = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 3 & -1 \\ -4 & 13 & -4 \end{pmatrix}$$

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, A\vec{v}_1 = \vec{v}_1 \text{ since eigenvector}$$

$$\text{Choose a vector in Ker}(A - 2I)^2 \text{ that is not in the previous one.}$$

$$\vec{v}_3 = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \vec{v}_2 = (A - 2I)\vec{v}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{cases} A\vec{v}_3 = \vec{v}_2 + 2\vec{v}_3 \text{ from } \vec{v}_2 = (A - 2I)\vec{v}_3 \\ A\vec{v}_2 = 2\vec{v}_2 \text{ by eigenvector} \end{cases}$$

$$\mathcal{B} = (\vec{v}_1, \vec{v}_2, \vec{v}_3)$$

$$\overbrace{A}^{[T]_{SS}} = \underbrace{\overbrace{\begin{pmatrix} 1 & -1 & 3 \\ 1 & 0 & 1 \\ 3 & 1 & 0 \end{pmatrix}}^{[I]_{S\mathcal{B}}}}_{P} \underbrace{\overbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}}^{[T]_{\mathcal{B}\mathcal{B}}}}_{J} \underbrace{\overbrace{\begin{pmatrix} 1 & -1 & 3 \\ 1 & 0 & 1 \\ 3 & 1 & 0 \end{pmatrix}^{-1}}^{[I]_{\mathcal{B}S}}}_{P^{-1}}$$

Motivation for exponentiating

$$A\vec{v}_1 = \lambda\vec{v}_1 \Rightarrow (A - \lambda I)\vec{v}_1 = \vec{0}$$

$$A\vec{v}_2 = \vec{v}_1 + \lambda\vec{v}_2 \Rightarrow (A - \lambda I)\vec{v}_2 = \vec{v}_1 \Rightarrow (A - \lambda I)(A - \lambda I)\vec{v}_2 = (A - \lambda I)\vec{v}_1 \Rightarrow (A - \lambda I)^2\vec{v}_2 = \vec{0}$$

Example

How many 5×5 nonsimilar matrices in Jordan form are there all of those whose eigenvalues are zero?

Solution

One block: 5×5 (1)
Two blocks: $1 \times 1 \wedge 4 \times 4 \vee 2 \times 2 \wedge 3 \times 3$ (2)
Three blocks: $1 \times 1 \wedge 1 \times 1 \wedge 3 \times 3 \vee 1 \times 1 \wedge 2 \times 2 \wedge 2 \times 2$ (2)
Four blocks: $1 \times 1 \wedge 1 \times 1 \wedge 1 \times 1 \wedge 2 \times 2$ (1)
Five blocks: $1 \times 1 \wedge 1 \times 1 \wedge 1 \times 1 \wedge 1 \times 1 \wedge 1 \times 1$ (1)

So 7 in total.

Example

Find the number of nonsimilar 6×6 matrices in Jordan form whose eigenvalues are $1, 2, 2, 3, 3, 3$.

Solution

For 1, 1 possibility (1×1). For 2, 2 possibilities ($1 \times 1 \wedge 1 \times 1 \vee 2 \times 2$. For 3, 3 possibilities ($1 \times 1 \wedge 1 \times 1 \wedge 1 \times 1 \vee 1 \times 1 \wedge 2 \times 2 \vee 3 \times 3$).

In total, $1 \times 2 \times 3 = 6$

Theorem: Finding J

Let $J \in M_n(\mathbb{C})$ and J be a matrix in Jordan form that is similar to A. Then, for every $k \in \mathbb{Z}^+$, the number of Jordan blocks of J with size at least $k \times k$ corresponding to an eigenvalue λ is $\dim \operatorname{Ker}(A - \lambda I)^k - \dim \operatorname{Ker}(A - \lambda I)^{k-1}$. Here $(A - \lambda I)^0 = I$

Example

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

$$\begin{cases} \dim \operatorname{Ker}\ (A - 2I) = 2 \\ \dim \operatorname{Ker}\ (A - 2I)^0 = 0 \end{cases} \rightarrow \text{There are } 2 - 0 \text{ Jordan blocks of size } 1 \times 1 \text{ or more}$$

$$\dim \operatorname{Ker}\ (A - 2I)^2 = \dim \operatorname{Ker} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 3 \therefore 3 - 2 = 1 \text{ block is bigger than } 2 \times 2$$

Example

The character polynomial of a matrix A is $p(z) = z^6 (z - 1)^4$. Suppose $\dim \operatorname{Ker} A = 1$ and $\dim \operatorname{Ker}\ (A - I) = 3$. Find a matrix in Jordan form that is similar to A.

Solution

For an eigenvalue of 0, we know there is a multiplicity of 6.

Because $\dim \operatorname{Ker}\,(A-0I)-\dim \operatorname{Ker}\,(A-0I)^0=1-0=1$. Therefore, there is one Jordan block. Since the multiplicity is 6, this Jordan block is 6×6

For an eigenvalue of 1, we know there is a multiplicity of 4. Because $\dim \operatorname{Ker}\,(A-1I)-\dim \operatorname{Ker}\,(A-1I)^0=3-0=3$. Therefore, there are 3 Jordan blocks. Since the multiplicity is 4, they have sizes 2×2, 1×1, and 1×1.

$$A=\begin{pmatrix} 0&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&1&1\\ 0&0&0&0&0&0&0&0&0&1 \end{pmatrix}$$

Nilpotent

For a matrix $A\in M_n(\mathbb{F})$, $\exists k A^k=0$

Applications of Jordan Form

$$A=P\begin{pmatrix} B_1 & & 0\\ & \ddots & \\ 0 & & B_m \end{pmatrix}P^{-1};A\in M_n(\mathbb{C})$$

$$B_j=\begin{pmatrix} \lambda_j & 1 & & 0\\ & \ddots & \ddots & \\ & & \ddots & 1\\ 0 & & & \lambda_j \end{pmatrix}=\begin{pmatrix} \lambda_j & & 0\\ & \ddots & \\ 0 & & \lambda_j \end{pmatrix}+\begin{pmatrix} 0 & 1 & & 0\\ & \ddots & \ddots & \\ & & \ddots & 1\\ 0 & & & 0 \end{pmatrix}=\lambda_j I+N_j$$

$$N_j\vec{e_1}=\vec{0},...,N_j\vec{e_k}=\overrightarrow{e_{k-1}}\therefore N_j^k=0\therefore N_j \text{ is nilpotent.}$$

$$A=\underbrace{P\begin{pmatrix} \lambda_1 I & & 0\\ & \ddots & \\ 0 & & \lambda_m I \end{pmatrix}P^{-1}}_{\text{Diagonalizable}}+\underbrace{P\begin{pmatrix} N_1 & & 0\\ & \ddots & \\ 0 & & N_m \end{pmatrix}P^{-1}}_{\text{Nilpotent}}$$

For every $A\in M_n(\mathbb{C})$, there are matrices $D,N\in M_n(\mathbb{C})$ such that.

1. $A=D+N$
2. D is diagonalizable.
3. N is nilpotent
4. $ND=DN$

Overall Objective

$$e^A=\sum_{m=0}^{\infty}\frac{A^m}{m!}$$

Show that e^A converges $\forall A \in M_n(\mathbb{C})$

A = D+N

Analytic function

A function $f : \mathbb{R} \to \mathbb{R}$ is called analytic if

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \;\; \forall x \in \mathbb{R}$$

$$a_n = \frac{f^{(n)}(0)}{n!}$$

Theorem: analytic over reals to convergent over the complex

If $\sum_{n=0}^{\infty} a_n x^n$ converges $\forall x \in \mathbb{R}$, then $\sum_{n=0}^{\infty} a_n z^n = 0$ converges $\forall z \in \mathbb{C}$.

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \forall z \in \mathbb{C} \text{ if } f \text{ is analytic}$$

Theorem: analytic function of a diagonalizable matrix

If $A = S\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} S^{-1}$, and f is analytic, then

$$f(A) = S\begin{pmatrix} f(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & f(\lambda_n) \end{pmatrix} S^{-1}$$

Proof

$$\text{Suppose } A = S\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} S^{-1}$$

$$f(A) = \sum_{m=0}^{\infty} a_m \left(S\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} S^{-1} \right)^m = \sum_{m=0}^{\infty} a_m S\begin{pmatrix} \lambda_1^m & & 0 \\ & \ddots & \\ 0 & & \lambda_n^m \end{pmatrix} S^{-1}$$

$$\text{Partial sum : } P_k(A) = \sum_{m=0}^{k} a_m S\begin{pmatrix} \lambda_1^m & & 0 \\ & \ddots & \\ 0 & & \lambda_n^m \end{pmatrix} S^{-1} = S\left(\sum_{m=0}^{k} a_m \begin{pmatrix} \lambda_1^m & & 0 \\ & \ddots & \\ 0 & & \lambda_n^m \end{pmatrix} \right) S^{-1} =$$

$$S\begin{pmatrix} \sum_{m=0}^{k} a_m \lambda_1^m & & 0 \\ & \ddots & \\ 0 & & \sum_{m=0}^{k} a_m \lambda_n^m \end{pmatrix} S^{-1}$$

$$f(A) = \lim_{k\to\infty} P_k(A) = \lim_{k\to\infty} S\begin{pmatrix} \sum_{m=0}^{k} a_m \lambda_1^m & & 0 \\ & \ddots & \\ 0 & & \sum_{m=0}^{k} a_m \lambda_n^m \end{pmatrix} S^{-1} = S\begin{pmatrix} f(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & f(\lambda_n) \end{pmatrix} S^{-1}$$

Theorem: analytic function of a diagonalizable matirx.

$$f(A) \text{ converges } \forall A \in M_n(\mathbb{C})$$

Proof

$$A = D + N$$

$$P_k(x) = \sum_{m=0}^{k} a_m x^m$$

$$P_k(x_0 + h) = \sum_{m=0}^{\infty} \frac{P_k^{(m)}(x_0)}{m!} h^m \qquad \text{Note that } P_k^{(m)} = 0 \forall m > \deg P_k \therefore \text{finite sum}$$

$$\text{Since } DN = ND, x_0 = D, h = N \text{ to obtain:}$$

$$P_k(D + N) = \sum_{m=0}^{\infty} \frac{P_k^{(m)}(D)}{m!} N^m.$$

$$\text{Suppose } N^l = 0, \text{which is true for some } l \text{ because } N \text{ is nilpotent.}$$

$$P_k(A) = \sum_{m=0}^{l-1} \frac{P_k^{(m)}(D)}{m!} N^m.$$

$$\text{Note } D \text{ is diagonalizable} \therefore \lim_{k\to\infty} P_k(D) = f(D) \wedge \lim_{k\to\infty} P_k^{(m)}(D) = f^{(m)}(D)$$

$$\therefore f(A) = \sum_{m=0}^{l-1} \frac{f^{(m)}(D)}{m!} N^m$$

Example

If $f(x) = e^x$, then

$$e^A = \sum_{m=0}^{l-1} \frac{e^D}{m!} N^m$$

$$e^A = e^D \sum_{m=0}^{l-1} \frac{1}{m!} N^m$$

$$e^A = e^D e^N$$

Example

Evaluate e^A and $\sin(B)$ where $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$.

Solution

Jordan Decomposition:

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, N^2 = 0$$

$$e^A = e^D \sum_{m=0}^{1} \frac{N^m}{m!} = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix} (I + N) = \begin{pmatrix} e & e \\ 0 & e \end{pmatrix}$$

For $\sin(B)$:

$$\det(B - zI) = (1-z)(-1-z) - 1 \cdot 3 = z^2 - 4 \Rightarrow z = \pm 2$$

$$z = 2 : \begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x = y \Rightarrow \left(2, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) \text{ is an eigenpair}$$

$$z = -2 : \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow y = -3x \Rightarrow \left(-2, \begin{pmatrix} 1 \\ -3 \end{pmatrix}\right) \text{ is an eigenpair}$$

$$B = \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}^{-1}$$

$$\sin(B) = \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} \sin 2 & 0 \\ 0 & \sin(-2) \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}^{-1}$$

Chapter 4: Ordinary Differential Equations

Definition of Differential Equation

An equation involving derivatives.

Definition of Ordinary Differential Equation

Equations involving only single-variable derivatives (not partial).

Example

$$\frac{d^2x}{dt^2} y + yx^2 \frac{dy}{dt} = t^2 + 3t - e^{t \sin t} : \text{ODE of order 2}$$

$$\left(\frac{\partial x}{\partial t}\right)^3 x(t,s) + ts - \sin t = x(t,s) : \text{PDE of order 1}$$

Definition of Partial Differential Equations

Equations involving partial derivatives.

Definition of Order

The highest derivative that appears in the differential equation.

Goals

1. Can we find a general solution to the differential equation?
2. Given initial values, is there a solution? If so, how many?
3. Can we approximate solutions given some initial values?
4. What are constant solutions?
5. What are periodic solutions?
6. What are bounded solutions?
7. Are all solutions bounded?
8. Are all solutions periodic?
9. Are all solutions odd?
10. Are all solutions even?
11. How do solutions change when initial values change?

Definition of Explicit Differential equations

Any differential equation of the form $\frac{dy}{dt} = f(t)$ is called an explicit differential equation.

Example

Find a general solution to $\frac{dy}{dt} = \frac{1}{t^2-t}$.

$$\frac{1}{t^2 - t} = \frac{1}{t(t-1)} = -\frac{1}{t} + \frac{1}{t-1}$$

$$\int \frac{1}{t-1} - \frac{1}{t} dt = \log(t-1) - \log(t) + \mathbb{c}_1 = \log\left(\frac{t-1}{t}\right) + \mathbb{c}_1, \mathbb{c}_1 \in \mathbb{C}$$

y is differential over $(0,1)$ and $y\left(\frac{1}{2}\right) = 5$ becomes

$$\log\left(\frac{\frac{1}{2} - 1}{\frac{1}{2}}\right) + \mathbb{c}_1 = 5 \Rightarrow \log(-1) + \mathbb{c}_1 = 5 \Rightarrow \mathbb{c}_1 = 5 - \log(-1) \Rightarrow$$

$$y = \log\left(\frac{t-1}{t}\right) + 5 - \log(-1) \Rightarrow y = \log\left(\frac{1-t}{t}\right) + 5$$

Existence and Uniqueness Theorem for explicit differential equations

Suppose $f : (a,b) \to \mathbb{R}$ is continuous. Then $\forall t_0 \in (a,b) \forall y_0 \in \mathbb{R}$ the initial value problem (IVP) given below has a unique solution.

$$\begin{cases} \frac{\mathrm{d}y}{\mathrm{d}t} = f(t) \\ y(t_0) = y_0 \end{cases}$$

Proof

Existence:

$$y(t) = \int_{t_0}^{t} f(u) \,\mathrm{d}u + y_0$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t) \qquad \text{by the fundamental theorem of calculus}$$

$$y(t_0) = \int_{t_0}^{t_0} f(u) \,\mathrm{d}u + y_0 = y_0$$

Uniqueness:

Assume y, z both satisfy the given IVP.

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t) = \frac{\mathrm{d}z}{\mathrm{d}z} \text{ and } y(t_0) = z(t_0)$$

$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}[y - z] = 0 \Rightarrow (y - z)(t) = \mathbb{c} \text{ Since } (y - z)(t_0) = 0, c = 0 \Rightarrow y = z \square$$

Definition of Linear Differential Equations

$a_1, ..., a_n, f, y$ are functions of t. $a_1, ..., a_n$ are called coefficients. f is called forcing.

$$y^{(n)}(t) + a_n(t)y^{(n-1)}(t) + \cdots + a_2(t)y'(t) + a_1(t)y(t) = f(t)$$

is called a linear differential equation in normal form.

Note that $L = D^n + a_n(t)D^{n-1} + \cdots + a_2(t)D + a_1(t)$ is linear. ($D = \frac{\mathrm{d}}{\mathrm{d}t}$)

$$L(y_1 + y_2) = L(y_1) + L(y_2); L(cy_1) = cL(y_1)$$

An initial value problem is an equation by:

$$\begin{cases} L(y) = f(t) \\ y(t_0) = y_0 \\ \vdots \\ y^{(n-1)}(t_0) = y_{n-1} \end{cases}$$

where $t_0, y_0, ..., y_{n-1}$ are given constants.

Solution to First-Order Linear Differential Equation

To solve: $y' + a(t)y = b(t)$

1. Find some $A(t)$ such that $A'(t) = a(t)$
2. $\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{A(t)}y\right) = e^{A(t)}b(t)$
3. Then solve

Proof

To solve: $y' + a(t)y = b(t)$

$$\mu y' + \mu a(t)y = \mu b(t)$$
$$\mu' = \mu a(t)$$
$$\therefore \mu = e^{\int a(t)\,\mathrm{d}t}$$
$$(\mu = A)$$

Existence and Uniqueness of First Order Linear Differential Equation.

Suppose $a_1(t)$ and $f(t)$ are continuous over (a, b). Let $t_0 \in (a, b)$ and $y_0 \in \mathbb{R}$ Then the initial value problem:

$$\begin{cases} \frac{\mathrm{d}y}{\mathrm{d}t} + a_1(t)y = f(t) \\ y(t_0) = y_0 \end{cases}$$

has a unique solution over (a, b).

Example

Solve $y' + y = e^t$.

Solution 1

$$(y\mu)' = y'\mu + y\mu' \text{ it seems to be in the right form...}$$
$$\mu(y' + y = e^t)$$
$$\mu y' + \mu y = \mu e^t$$
$$\text{If } \mu = \mu' \text{ then we'd have product rule} \therefore \mu = e^t$$
$$y' + y = e^t \Rightarrow e^t y' + e^t y = e^{2t} \Rightarrow \left(ye^t\right)' = e^{2t} \overset{\text{explicit}}{\Rightarrow} ye^t = \frac{1}{2}e^{2t} + \mathbb{c} \Rightarrow y = \frac{1}{2}e^t + \mathbb{c}e^{-t} \ \square$$

Solution 2

Set L(y) = y' + y. L is linear. By a theorem, $L^{-1}(\{e^t\}) = y_1 + \text{Ker } L$ where y_1 is a solution to $y_1' + y_{1'} = e^t$.

$$y_1 = \frac{1}{2}e^t \text{ by guessing.}$$
$$y \in \text{Ker } L \Longleftrightarrow y' + y = 0 \Rightarrow y' = -y \Rightarrow \left(e^t y\right)' = 0 \Rightarrow e^t y = c \Rightarrow y = ce^{-t}$$
$$y = \frac{1}{2}e^t + ce^{-t}, c \in \mathbb{R} \ \square$$

Separable Equation

A separable differential equation is one of the form $\frac{\mathrm{d}y}{\mathrm{d}t} = f(t)g(y)$.

To solve, write as $\frac{\mathrm{d}y}{g(y)} = f(t)\,\mathrm{d}t$ and integrate.

$\int \frac{1}{g(y)}\,\mathrm{d}y = \int \frac{1}{g(y)}y'\,\mathrm{d}t = \int f(t)\,\mathrm{d}t$

Example

Solve $\frac{\mathrm{d}y}{\mathrm{d}t} = 2ty^2 + 3t^2y^2$. Can you find a solution that satisfies $y(1) = 0$?

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}t} = (2t + 3t^2)y^2 \Rightarrow \frac{\mathrm{d}y}{y^2} = (2t + 3t^2)\,\mathrm{d}t \Rightarrow$$
$$-\frac{1}{y} = t^2 + t^3 + \mathbb{c} \Rightarrow y = -\frac{1}{t^2 + t^3 + \mathbb{c}}$$
$$y(1) = 0 \Rightarrow -\frac{1}{2 + t} = 0 \times$$
$$y(t) = 0 \text{ is a solution}$$

Stationary Definition

A solution to a differential equation is called stationary, fixed, equilibrium, or a critical point if it is a constant.

Example

find all solutions of

$$\begin{cases} \frac{\mathrm{d}y}{\mathrm{d}t} = ty^2 - ty \\ y(1) = 2 \end{cases}$$

Solution

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}t} &= t(y^2 - y) \\ \log\left(\frac{y-1}{y}\right) &= \frac{t^2}{2} + \mathbb{c} \\ \Rightarrow \frac{y-1}{y} &= e^{\frac{t^2}{2}+\mathbb{c}} \\ \Rightarrow \frac{y-1}{y} &= \mathbb{c}e^{\frac{t^2}{2}}.\end{aligned}$$

Stationary
$y^2 - y = 0 \Rightarrow y = 0, 1$

$$\frac{y-1}{y} = \mathbb{c}e^{\frac{t^2}{2}}, \mathbb{c} \in \mathbb{R}$$

Change of variables

In general, to solve $y' = f(ay + bt + c)$ for constants $a, b, c \in \mathbb{R}$, we set $u = ay + bt + c$.

$$u' = ay' + b = af(u) + b \Rightarrow u' = af(u) + b$$

Which is separable.

Any DE of the form $y' = f(y)$ is called autonomous.

In general, to solve $y' = f\left(\frac{y}{t}\right)$, substitute $u = \frac{y}{t}$.

$$ut = y \Rightarrow u't + u = u' = f(u) \Rightarrow u't = f(u) - u$$

Some examples:

$$y' = \frac{ay + bt}{cy + dt} = \frac{a\left(\frac{y}{t}\right) + b}{c\left(\frac{y}{t}\right) + d}$$

$$y' = \frac{y^2 + 3yt + t^2}{3y^2 + 7yt} = \frac{\left(\frac{y}{t}\right)^2 + 3\left(\frac{y}{t}\right) + 1}{3\left(\frac{y}{t}\right)^2 + 7\left(\frac{y}{t}\right)}$$

In general, to solve $y' = \frac{ay+bt+c}{my+nt+k}$ with $a, b, c, m, n, k \in \mathbb{R}$, select constants $Y = y + r, T = t + s$ where it cancels out the constants and becomes like the above solution. If not possible, substitute something else.

Example
Solve:

$$y' = \frac{e^{y+t} - y - t}{y + t}$$

Solution
Define a function $u = y + t \Rightarrow u' = y' + 1$

$$u' - 1 = \frac{e^u - u}{u} \Rightarrow u' = \frac{e^u}{u}$$

Stationary: $\frac{e^u}{u} = 0$ has no solutions

$$\int \frac{u}{e^u} \, \mathrm{d}u = \int \mathrm{d}t$$

$$f = u, \mathrm{d}f = 1, \mathrm{d}g = e^{-t}, g = -e^{-t}$$

$$u = t + \mathbb{c}$$

$$-ue^{-u} - e^{-u} = t + \mathbb{c}$$

$$\boxed{-(y + t)e^{-y-t} - e^{-y-t} = t + c}$$

Example

Solve

$$y' = \frac{y - t}{y + t}$$

Solution

$$y' = \frac{\frac{y}{t} - 1}{\frac{y}{t} + 1}$$

$$u = \frac{y}{t} \Rightarrow ut = y \Rightarrow u't + u = y' \Rightarrow \frac{u - 1}{u + 1} \Rightarrow u't = \frac{u - 1}{u + 1} - u = \frac{-1 - u^2}{u + 1} \Rightarrow u' = \frac{-1 - u^2}{u + 1} \cdot \frac{1}{t}$$

$$\text{which is separable}$$

$$\frac{-1 - u^2}{u + 1} = 0 \Rightarrow 1 + u^2 = 0 \Rightarrow \text{no stationary solution} \int \frac{u + 1}{u^2 + 1} \, \mathrm{d}u = \int -\frac{1}{t} \, \mathrm{d}t$$

$$\int \frac{u}{u^2 + 1} \, \mathrm{d}u + \int \frac{1}{u^2 + 1} \, \mathrm{d}u = \log(t) + \mathbb{c}$$

$$\frac{1}{2} \log(u^2 + 1) + \arctan(u) = \log(t) + \mathbb{c}$$

$$\boxed{\frac{1}{2} \log\left(\left(\frac{y}{t}\right)^2 + 1\right) + \arctan\left(\frac{y}{t}\right) = \log(t) + \mathbb{c}}$$

Example

Solve $y' = \frac{y-t+1}{y+t-3}$

Solution

$$T = t + a, Y = y + b \text{ for some } a, b \in \mathbb{R}.$$

$$\frac{Y - b - T + a + 1}{Y - b + T - a - 3} \to \begin{cases} -b + a + 1 = 0 \\ -b - a - 3 = 0 \end{cases} \to \begin{cases} b = -1 \\ a = -2 \end{cases} \to \frac{\mathrm{d}Y}{\mathrm{d}T} = \frac{Y - T}{Y + T}, T = t - 2, Y = y - 1$$

$$\vdots \text{ look above for the answer to this}$$

$$\boxed{\frac{1}{2}\log\left(\left(\frac{y-1}{t-2}\right)^2 + 1\right) + \arctan\left(\frac{y-1}{t-2}\right) = \log(t-2) + \mathbb{c}}$$

$$(\text{assume } \mathbb{c} \in \mathbb{C} \text{ and the complex logarithm})$$

Exact Equations and Integrating Factor

Suppose $\Phi(t, y) = c$ solves a differential equation.

$$\Phi_t + \Phi_y y' = 0 \text{ by chain rule}$$

$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}(\Phi(t, y(t))) \Rightarrow \Phi(t, y) = c$$

Example

Solve $e^t y + 2t + (2y + e^t)y' = 0$

Solution

Let's find Φ

$$\begin{cases} \int \Phi_t \,\mathrm{d}t = \int e^t y + 2t \,\mathrm{d}t \\ \int \Phi_y \,\mathrm{d}y = \int 2y + e^t \,\mathrm{d}y \end{cases} \to \begin{cases} \Phi = e^t y + t^2 + f(y) \\ \Phi = e^t y + y^2 + g(t) \end{cases} \overset{\text{Let } f(y)=y^2 \wedge g(t)=t^2}{\to} \Phi = e^t y + t^2 + y^2 \therefore \text{solutions are } e^t y + t^2 + y^2 = c, c \in \mathbb{R}$$

Exact Definition

A differential equation $M(t, y) + N(t, y)\frac{\mathrm{d}y}{\mathrm{d}(t)} = 0$ is called exact over an open rectangle $R = (a, b) \times (c, d)$ on the ty-plane if there is a function $\Phi(t, y)$ defined on R such that $\Phi_t = M$ and $\Phi_y = N$ for all $(t, y) \in R$. Sometimes this is written as $M(t, y)\,\mathrm{d}t + N(t, y)\,\mathrm{d}y = 0$.

$$\begin{cases} \Phi_t = M \\ \Phi_y = N \end{cases}$$

$$\Rightarrow \vec{f} = M\vec{i} + N\vec{j} + 0\vec{k} \Rightarrow \operatorname{curl} \vec{F} = (N_t - M_y)\vec{k}$$

Theorem for seeing if Exact

Let $M(t, y), N(t, y)$ be C^1 functions over an open rectangle $R = (a, b) \times (c, d)$ on the ty-plane. Then, there is a function $\Phi(t, y)$ defined over R such that $\Phi_t = M$ and $\Phi_y = N$ over R iff $M_y = N_t$

Proof

$\Rightarrow$ Suppose $\Phi_t = M \wedge \Phi_y = N$ then $M_y = \Phi_{ty} = \Phi_{yt} = N_t$ by Clairaut's Theorem. Notice $M_y \wedge N_t$ are both continuous by being C^1.

$\Leftarrow$ Suppose $M_y = N_t$ we will find Φ such that

$$\begin{cases} \Phi_t(t,y) = M(t,y) \\ \Phi_y(t,y) = N(t,y) \end{cases}$$

$$\text{Fix } t_0 \text{ inside } (a,b)$$

$$\Phi(t,y) = \int_{t_0}^{t} M(x,y)\,\mathrm{d}x + g(y)$$

$$\Phi_y = \int_{t_0}^{t} M_y(x,y)\,\mathrm{d}x + g'(y)$$

$$= \int_{t_0}^{t} N_x(x,y)\,\mathrm{d}x + g'(y)$$

$$= N(x,y)|_{x=t_0}^{x=t} + g'(y) = N(t,y) - N(t_0,y) + g'(y)$$

$$\Phi_y = 0 \iff N(t_0,y) = g'(y). \text{ Since } N \text{ is continuous and depends only on y,}$$

$$g(y) = \int N(t_0,y)\,\mathrm{d}y \text{ exists as a function of } y$$

Example

Solve $(xy^2 + y + e^x) + (x^2y + x)y' = 0$

Is this exact?

$$\frac{\mathrm{d}}{\mathrm{d}y}[xy^2 + y + e^x] = 2yx + 1$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[x^2y + x] = 2yx + 1$$

$$\text{Yes it is exact.}$$

$$\Phi_x = xy^2 + y + e^x \Rightarrow \Phi = \int xy^2 + y + e^x\,\mathrm{d}x = \frac{x^2y^2}{2} + xy + e^x + f(y)$$

$$\Phi_y = x^2y + x \Rightarrow x^2y + x + f'(y) = xy^2 + x \Longrightarrow f'(y) = 0 \Rightarrow f(y) = 0 \text{ works}$$

$$\therefore \boxed{\frac{x^2y^2}{2} + xy + e^x = \mathbb{c}}$$

Example

Solve the IVP:

$$\begin{cases} 3t^2y + 8ty^2 + (t^3 + 8t^2y + 12y^2)y' = 0 \\ y(2) = 1 \end{cases}$$

Solution

$$\frac{\partial}{\partial y}(3t^2y+8ty^2) = 3t^2+16ty = \frac{\partial}{\partial t}(t^3+8t^2y+12y^2) \therefore \text{exact}$$

$$\Phi_t = 3t^2y+8ty^2 \Rightarrow \Phi = t^3y+4t^2y^2+g(y)$$

$$\Phi_y = t^3+8t^2y+12y^2 \Rightarrow t^3+8t^2y+g'(y) = t^3+8t^2y+12y^2 \Rightarrow g'(y) = 12y^2 \Rightarrow g(y) = 4y^3 \text{ works}$$

$$\therefore t^3y+4t^2y^2+4y^3 = \mathbb{c} \text{ are all the solutions}$$

$$2^3(1)+4(2)^2(1)^2+4(1)^3 = \mathbb{c}$$

$$8+16+4 = 28 = \mathbb{c}$$

$$\therefore \boxed{t^3y+4t^2y^2+4y^3 = 28}$$

Example
Solve

$$2ty+(2t^2-e^y)y' = 0$$

.

Solution

$$\frac{\partial}{\partial y}2ty = 2t \neq 4t = \frac{\partial}{\partial t}(2t^2-e^y)$$

Exact does not work.

$$2ty+(2t^2-e^y)y' = 0$$

$$2t\mu y+(2t^2-e^y)\mu y' = 0$$

$$\text{We need to find } \mu \text{ so the differential equation is exact.}$$

$$\frac{\partial}{\partial y}2ty\mu = \frac{\partial}{\partial t}(2t^2-e^y)\mu$$

$$2t\mu+2ty\mu_y = 4t\mu+(2t^2-e^y)\mu_t$$

$$\mu = y \text{ because we need to kill the } e^y$$

$$2t(y)+2ty(1) = 4t(y)+(2t^2-e^y)\cdot 0$$

$$4ty = 4ty$$

$$\therefore \text{now exact}$$

$$\vdots$$

$$e^y(y-1)+t^2y^2 = \mathbb{c}$$

To find an integrating factor
1. multiply both sides by μ
2. use the $M_y = N_t$ to find a PDE in terms of μ
3. Try $\mu = \mu(t)$ and $\mu = \mu(y)$

Example
Solve $4xy+3y^3+(x^3+3xy^2)y' = 0$

Solution

$$\frac{\partial}{\partial y}(4xy+3y^3)=4x+9y^2\neq 2x+3y^2=\frac{\partial}{\partial x}(x^2+3xy^2)\ \therefore \text{not exact}$$

$$\frac{\partial}{\partial y}((4xy+3y^3)\mu)=\frac{\partial}{\partial x}((x^2+3xy^2)\mu)$$

$$(4x+9y^2)\mu+(4xy+3y^3)\mu_y=(2x+3y^2)\mu+(x^2+3xy^2)\mu_x$$

$$(2x+6y^2)\mu+(4xy+3y^3)\mu_y=(x^2+3xy^2)\mu_x$$

$$\mu=x^2$$

$$\vdots$$

$$y^3x^3+yx^4=\mathbb{c}$$

Chapter 5: Existence and Uniqueness Theorems

Usually used when you can't actually solve a differential equation.

Existence and Uniqueness Theorem for Linear Equations

Let I be an open interval, and let $a_1(t), ..., a_{n(t)}, f(t)$ be continuous functions over I. For every $t_0 \in I$ and every $y_0, ..., y_{n-1} \in \mathbb{R}$ the following initial value problem has a unique solution:

$$\begin{cases} y^{(n)}+a_n(t)y^{(n-1)}+\cdots+a_2(t)y'+a_1(t)y=f(t) \\ y(t_0)=y_0 \\ y'(t_0)=y_1 \\ \vdots \\ y^{n-1}(t_0)=y_{n-1} \end{cases}$$

$$D=\frac{\mathrm{d}}{\mathrm{d}t}; L=D^n+a_{n(t)}D^{n-1}+\cdots+a_2(t)D+a_1(t)$$

$$L(y)=f(t).$$

Example

Find the largest interval I for which the existence and uniqueness theorem guarantees a unique solution to the IVP for $t \in I$.

$$\begin{cases} ty''+\frac{\tan(t)}{t-3}y'-y=e^t \\ y(1)=2 \\ y'(1)=4 \end{cases}$$

Solution

$$y'' + \frac{\tan(t)}{t(t-3)}y' + \frac{1}{t}y = \frac{e^t}{t}$$

$$t_0 = 1 \therefore 1 \in I$$

$$\tan(t) \Rightarrow t \neq \frac{\pi}{2} + k\pi, k \in \mathbb{Z}$$

$$t(t-3) \Rightarrow t \neq \{0, 3\}$$

$$\text{Because } 1 \in I, I = \left(0, \frac{\pi}{2}\right)$$

Example

Prove the function $y = \sin(t^2)$ cannot be a solution to a second-order homogeneous linear differential equation whose coefficients are continuous over $(-1, 1)$.

Solution 1

Suppose on the contrary $y = \sin(t^2)$ satisfies

$$y'' + a_2(t)y' + a_1(t)y = 0 \text{ with } a_1, a_2 \text{ continuous over } (-1, 1)$$

$$y' = 2t\cos(t^2)$$

$$y'' = 2\cos(t^2) - 4t^2\sin(t^2)$$

$$2\cos(t^2) - 4t^2\sin(t^2) + a_2(t)2t\cos(t^2)y' + a_1(t)y = 0$$

$$2\cos(t^2) - 4t^2\sin(t^2) + a_2(t)2t\cos(t^2) + a_1(t)\sin(t^2) = 0$$

$$\text{Plug in } 0$$

$$2 - 0 + 0 + 0 \stackrel{?}{=} 0$$

$$2 \neq 0 \therefore \text{cannot be a solution}$$

Solution 2

$$y = \sin(t^2)$$

$$y' = 2t\cos(t^2)$$

$$\begin{cases} y'' + a_2(t)y' + a_1(t)y = 0 \\ y(0) = 0 \\ y'(0) = 0 \end{cases}$$

$$y(t) = 0 \text{ is another solution} \therefore \text{by uniqueness } y = \sin(t^2) \text{ cannot be a solution.}$$

Picard Iterates

To show a solution to the IVP below exists:

$$\begin{cases} \frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y) \\ y(t_0) = y_0 \end{cases}$$

We:

1. Create a sequence y_n of functions.
2. Show y_n converges to y.

3. Show y satisfies the IVP.

Let y_0 be the first term, a constant.

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t,y) \Rightarrow y = \int f(t,y(t))\mathrm{d}t$$

$$\Rightarrow y = y_0 + \int_{t_0}^{t} f(s,y(s))\mathrm{d}s$$

$$y_1(t) = y_0 + \int_{t_0}^{t} f(s,y_0)\mathrm{d}s$$

$$y_2(t) = y_0 + \int_{t_0}^{t} f(s,y_1(s))\mathrm{d}s$$

$$\vdots$$

$$y_{n+1}(t) = y_0 + \int_{t_0}^{t} f(s,y_n(s))\mathrm{d}s$$

$y_0, y_1, y_2, \ldots$ are called Picard Iterates

Proofish

$$\text{Assume } y_n \to y$$

$$y_n = y_0 + \int_{t_0}^{t} f(s,y_{n-1}(s))\mathrm{d}s$$

$$\downarrow$$

$$y = y_0 + \int_{t_0}^{t} f(s,y(s))\mathrm{d}s$$

$$\downarrow$$

$$y' = f(t,y(t))$$

Example
Solve the IVP $y' = y^2, y(0) = 1$. Show this solution is not defined over $\mathbb{R}$.

Solution
This is separable. Since $y(0) \neq 0$, the solution is not stationary.

$$\int \frac{\mathrm{d}y}{y^2} = \int 1\mathrm{d}t \Rightarrow -\frac{1}{y} = t + \mathbb{c}_1 \overset{y(0)=1}{\Rightarrow} y = -\frac{1}{t+\mathbb{c}_1} \Rightarrow y(t) = -\frac{1}{t-1}$$

The largest possible domain of this solution is $(-\infty, 1) \neq \mathbb{R}$

Example
Find two solutions to the IVP

$$y' = 3y^{\frac{2}{3}}, y(0) = 0$$

Solution

$$y = 0 \text{ is one solution by inspection}$$
$$\int \frac{y'}{3y^{\frac{2}{3}}} \mathrm{d}t = \int 1 \mathrm{d}t$$
$$\int \frac{1}{3y^{\frac{2}{3}}} = t + \mathbb{c}_1$$
$$y^{\frac{1}{3}} = t + \mathbb{c}_1$$
$$y = (t + \mathbb{c}_1)^3$$
$$\therefore y = t^3 \text{ is a solution}$$

All solutions:

$$\forall a \leq 0 \leq b \in \mathbb{R}, y = \begin{cases} (t-b)^3 \text{ if } t > b \\ 0 \text{ if } a \leq t \leq b \\ (t-a)^3 \text{ if } t < a \end{cases}$$

Thoughts

$$\begin{cases} \frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y) \\ y(t_0) = y_0 \end{cases}$$

How do we show the existence of something without finding it?

Examples of knowing the existence but not finding it

- Definite integrals are defined as a limit of Riemann sums.

We need to show that the limit exists

- Initial value problem of linear differential equations
- Summation converging

Example of Summation converging

Prove $f(x) = \sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2}$ is defined for all $x \in \mathbb{R}$.

Solution

$$-1 \leq \sin(nx) \leq 1 \Rightarrow \frac{|\sin(nx)|}{n^2} \leq \frac{1}{n^2}$$
$$\text{By } p\text{-test with } p = 2, 2 > 1, \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges.}$$
$$\text{By the comparison test, } \sum_{n=1}^{\infty} \frac{|\sin(nx)|}{n^2} \text{ converges.}$$
$$\text{By the absolute convergence test, } \sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2} \text{ converges.}$$

We know the above theorems by taking the limit of partial sums; let's try a similar idea with an IVP.

Example

Compute the Picard Iterates for the IVP

$$\begin{cases} y' = \underbrace{y}_{f(t,y)} \\ y(\underbrace{0}_{t_0}) = \underbrace{1}_{y_0} \end{cases}$$

and show they converge to the solution this given IVP.

Solution

$$y_0 = 1$$
$$y_1 = 1 + \int_0^t 1 \mathrm{d}s = 1 + t y_2 = 1 + \int_0^t 1 + s \mathrm{d}s = 1 + t + \frac{t^2}{2}$$
$$y_3 = 1 + \int_0^t 1 + s + \frac{s^2}{2} \mathrm{d}s = 1 + t + \frac{t^2}{2} + \frac{t^3}{3!}$$
$$\vdots$$
$$\text{Claim: } y_n = \sum_{k=0}^{n} \frac{t^k}{k!}$$
$$\text{Proof by induction on } n.$$
$$\text{Base case } n = 0 \to y_0 = 1 \checkmark$$
$$\text{Inductive Hypothesis : Suppose } y_n = \sum_{k=0}^{n} \frac{t^k}{k!}$$
$$\text{Inductive Step:}$$
$$y_{n+1} = 1 + \int_0^t \sum_{k=0}^{n} \frac{s^k}{k!} \mathrm{d}s = \sum_{k=0}^{n+1} \frac{t^k}{k!}$$
$$\lim_{n\to\infty} y_{n(t)} = \sum_{k=0}^{\infty} \frac{t^k}{k!} = e^t$$

$$y' = y \to \frac{y'}{y} = 1 \to \int \frac{y'}{y} \mathrm{d}t = \int 1 \mathrm{d}t \to \log(y) = t + \mathbb{c} \to y = e^{t+\mathbb{c}} \to y = \mathbb{c}e^t \to y = e^t$$
$$e^t = e^t, \text{ which is what we got via picard iterates.}$$

Integral Comparison Theorems

$$\forall x \in [a,b] f(x) \le g(x) \Rightarrow \int_a^b f(x) \mathrm{d}x - \int_a^b g(x) \mathrm{d}x$$
$$\left| \int f(x) \mathrm{d}x \right| \le \int |f(x)| \mathrm{d}x$$

Mean Value Theorem

$$\exists c \in [a,b] f'(c) = \frac{f(b) - f(a)}{b - a}$$

First-Order Differential Equations Existence and Uniqueness

Suppose $f(t,y)$ and $f_y(t,y)$ are continuous over a rectangle R on the ty-plane given by $t_0 \leq t \leq t_0 + a$, $|y - y_0| \leq b$. Let M be $\forall (t,y) \in R, |f(t,y)| \leq M$ (this must exist by the extreme value theorem) and let $\alpha = \min\left(a, \frac{b}{M}\right)$. Then the IVP

$$\begin{cases} \frac{\mathrm{d}y}{\mathrm{d}t} = f(t,y) \\ y(t_0) = y_0 \end{cases}$$

has a continuous solution defined over $[t, t_0 + \alpha]$. Furthermore, $|y(t) - y_0| \leq b$ for all $t \in [t_0, t_0 + \alpha]$.

A similar result holds for $[t_0 - \alpha, t_0]$ and $[t_0 - \alpha, t_0 + \alpha]$

Proofish of existence

$$M \text{ is the maximum of } |f(t,y)|$$

$$\therefore -M \leq \frac{\mathrm{d}y}{\mathrm{d}t} \leq M$$

$$\therefore \text{the max or min } y \text{ can reach is } y = Mt + y_0 \vee y = -Mt + y_0 \text{ for } t \in R.$$

$$\text{Solve for escape: } y_0 + b = Mt + y_0 \vee y_0 - b = -Mt + y_0 \Rightarrow b = Mt \vee -b = -Mt \Rightarrow t = \frac{b}{M} \therefore \text{the solution would exist for } t \in \left[t_0, \frac{b}{M}\right] \cap \underbrace{R}_{\text{where the maximum slope is guaranteed over}}.$$

Example

Consider the IVP

$$\begin{cases} \frac{\mathrm{d}y}{\mathrm{d}t} = t + e^{-y^2} \\ y(0) = 0 \end{cases}$$

Show there is a unique solution defined over $\left[0, \frac{1}{2}\right]$ and that this solution satisfies $|y(t)| \leq 1$ for all $t \in \left[0, \frac{1}{2}\right]$.

Solution

$$t_0 = 0, y_0, \text{Suppose } a, b > 0, \text{and } t \in [0, a], y \in [-b, b]$$

$$\left|t + e^{-y^2}\right| \leq a + e^{-0} = a + 1$$

$$M = a + 1, \alpha = \min\left(a, \frac{b}{a+1}\right), t + e^{-y^2} \text{ is } C^\infty. \text{ Thus, we can apply the theorem.}$$

$$\therefore \exists \text{ a unique solution } y(t) \text{ defined over } [0, \alpha]$$

$$\text{Let } a = 1, b = 1.$$

$$\alpha = \min\left(a, \frac{b}{a+1}\right) \min\left(1, \frac{1}{1+1}\right) = \min\left(1, \frac{1}{2}\right) = \frac{1}{2}.$$

$$|y| \leq b \Rightarrow |y| \leq 1$$

$$\therefore \text{ a unique solution is defined over } \left[0, \frac{1}{2}\right] \text{ and } |y| \leq 1$$

Proof of Uniqueness

Suppose $z_1(t)$ and $z_2(t)$ both satisfy the given IVP: $\begin{cases} \frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y) \\ y(t_0) = y_0 \end{cases}$

$$\text{Recall } y(t) = y_0 \int_{t_0}^{t} f(s, y(s))\mathrm{d}s$$

$$|z_1(t) - z_2(t)| =$$

$$\left| y_0 + \int_{t_0}^{t} f(s, z_1(s))\mathrm{d}s - y_0 - \int_{t_0}^{t} f(s, z_2(s))\mathrm{d}s \right|$$

$$= \left| \int_{t_0}^{t} f(s, z_1(s)) - f(s, z_2(s))\mathrm{d}s \right|$$

$$\leq \int_{t_0}^{t} |f(s, z_1(s)) - f(s, z_2(s))|\mathrm{d}s$$

f_y exists and f is continuous. By the MVT there is some c between $z_1(s)$ and $z_2(s)$ such that

$$f(s, z_1(s)) - f(s, z_2(s)) = f_y(s, c)(z_1(s) - z_2(s))$$

$$\Rightarrow |z_1(t) - z_2(t)| \leq \int_{t_0}^{t} |f_y(s, c)||(z_1(s) - z_2(s))|\mathrm{d}s$$

Since f_y is continuous over R, by the EVT, there is a constant L such that $|f_y(s, c)\ | \leq L,$ for all s, c

$$\Rightarrow \underbrace{|z_1(t) - z_2(t)|}_{W'(t)} \leq L \underbrace{\int_{t_0}^{t} |(z_1(s) - z_2(s))|\mathrm{d}s}_{W(t)}$$

$$\Rightarrow W'(t) - LW(t) \leq 0$$

$$\Rightarrow e^{-Lt}W'(t) - Le^{-Lt}W(t) \leq 0$$

$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}(e^{-Lt}W(t)) \leq 0$$

$$\therefore e^{-Lt}W(t) \text{ is decreasing over } [t_0, t_0 + \alpha].$$

$$\Rightarrow e^{-Lt}W(t) \leq e^{-Lt_0}W(t_0) = e^{-Lt_0} \int_{t_0}^{t_0} |z_1(s) - z_2(s)|\mathrm{d}s = 0$$

$$\Rightarrow W(t) \leq 0 \Rightarrow \int_{t_0}^{t} |z_1(s) - z_2(s)|\ \mathrm{d}s \leq 0$$

$$|z_1(s) - z_2(s)| \geq 0 \Rightarrow \int_{t_0}^{t} |z_1(s) - z_2(s)|\mathrm{d}s \geq 0$$

$$\Rightarrow \int_{t_0}^{t} |z_1(s) - z_2(s)| = 0 \overset{\text{differentiation}}{\Longrightarrow} |z_1(t) - z_2(t)| = 0 \Rightarrow z_1 = z_2$$

$\mathcal{B}$

Example

Show the following IVP has a unique solution defined over $[0, \infty)$:

$$\begin{cases} \frac{\mathrm{d}y}{\mathrm{d}t} = e^{-y^2} + t^2 \\ y(0) = 1 \end{cases}$$

Solution

To show infinity, show that it can go to any arbitrary number h.

$$\text{Let } a, b > 0, 0 \leq t \leq a, |y - 1| \leq b$$
$$e^{-y^2} + t^2 \text{ is } C^\infty \therefore C^1$$
$$|e^{-y^2} + t^2| = e^{-y^2} + t^2 \leq 1 + t^2 \leq 1 + a^2 = M$$
$$\alpha = \min\left(a, \frac{b}{1 + a^2}\right)$$
$$\text{Let } b = a(1 + a^2) \Rightarrow \alpha = \min(a, a) = a$$
$$\text{By the existence and uniqueness theorem there is a solution over } [0, a] \text{ for all } a$$
$$\text{Let } y_n \text{ be the unique solution to the given IVP with } t \in [0, n] \forall n \in \mathbb{Z}^+.$$
$$\text{Note that if } n < m \text{ then } y_n(t) = y_m(t) \forall t \in [0, n].$$
$$\because y_n \wedge y_m \text{ both satisfy the given IVP with } t \in [0, n] \text{ and by uniqueness } y_n = y_m \text{ over } [0, n]$$
$$\text{Define } y(t) = y_n(t), 0 \leq t \leq n$$
$$y'(t) = y_n'(t) = e^{-y^2} + t^2, y(0) = y_n(0) = 1 \quad \text{(existence)}$$
$$\text{Uniqueness : Assume } y, z \text{ are both solutions. Use the uniqueness of } y_n \text{ to show } y = z \text{ over } \mathbb{R}$$

Chapter 6: Numerical Methods

$$\begin{cases} \frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y) \\ y(t_0) = y_0 \end{cases}$$

Sometimes, we can solve these, but that is uncommon in the space of possible functions. We do, however, know there is a solution by the existence and uniqueness theorem.

Assuming a unique solution exists, can we approximate $y(t_f)$?

The Lagrange Remainder Theorem

$$f(a + h) = f(a) + \frac{f'(a)}{1!}h + \frac{f''(a)}{1!}h^2 + \dots + \frac{f^{(n)}(a)}{1!}h^n + \frac{f^{(n+1)}(c)}{(n+1)!}h^{n+1}$$
$$c \in [a, a + h]$$

$$n = 0 \Rightarrow \underbrace{f(a + h) = f(a) + f'(c)h}_{\approx \text{ linear approximation}} \Rightarrow \underbrace{f'(c) = \frac{f(a + h) - f(a)}{h}}_{\text{Mean Value theorem}}$$

Euler's Method

To approximate $y\left(t_f\right)$ we divide $\left[t_0, t_f\right]$ into n subintervals of width $h = \frac{t_f - t_0}{n}$. Let $t_k = t_0 + kh$ for $k = 1, ..., n$

$$\begin{cases} y_0 = y_0 \\ y_k = y_{k-1} + hf(t_{k-1}, y_{k-1}) \end{cases} \Longrightarrow y_k \approx y(t_k)$$

Getting the idea

Example

Approximate $y' + y^2 + t^2, y(0) = 1$. Approximate $y(0.2)$ using 1 step then 2 steps

Solution

First, prove that there is a unique solution defined over $[0, 0.2]$.

$$\vdots$$

$$\therefore \square$$

1 Step

$$y(0.2) \approx 1 + 0.2\left(1^2 + 0^2\right) = 1.2 \Rightarrow y(0.2) \approx 1.2$$

2 Step

$$t_0 = 0, t_1 = 0.1, t_2 = 0.2, h = 0.1, y_0 = 1$$

$$y_1 = \underbrace{1}_{y_0} + \underbrace{0.1}_{h}\underbrace{\left(1^2 + 0^2\right)}_{f(0,1)} = 1.1$$

$$y_2 = \underbrace{1.1}_{y_1}\underbrace{0.1}_{h}\underbrace{\left((1.1)^2 + (0.1)^2\right)}_{f(0.1,1.1)} = 1.222$$

Euler's Method: Error

Objective: Find an approximation for $|y(t_n) - y_n|$.

$$\text{Use the Lagrange Remainder Theorem:}$$

$$\text{Set } n = 1 : f(a+h) = f(a) + f'(a)h + \frac{f''(c)}{2}h^2$$

$$\text{Substitute } a = 5_k, h = \text{step size} = \frac{t_n - t_0}{n}, f = y$$

$$f(t_k + h) = y(t_k) + y'(t_k)h + \frac{y''(\mathbb{c}_k)}{2}h^2; t_0 \leq \mathbb{c}_k \leq t_k + h$$

$$y' = f(t,y) \Rightarrow y'' = f_t + f_y \cdot y' \Rightarrow f_t + f_y f$$

$$y(t_{k+1}) = y(t_k) + f(t_k, y(t_k))h\frac{(f_t + f_y f)(\mathbb{c}_k)}{2}h^2$$

$$y(t_{k+1} - y_{k+1}) = y(t_k) - y_k + h(f(t_k, y(t_k)) - f(t_k, y_k)) + \frac{h^2}{2}(f_t + f_y f)(\mathbb{c}_k, y(\mathbb{c}_k))$$

$$\text{By the MVT as a function of y: } \frac{f(t_k, y(t_k)) - f(t_k, y_k)}{y(t_k) - y_k} = f_y(t_k, \mathbb{d}_k), \text{for some } \mathbb{d}_k \text{ between } y(t_k) \text{ and } y_k$$

$$\Rightarrow f(t_k, y(t_k)) - f(t_k, y_k) = (y(t_k) - y_k) f_y(t_k, \mathbb{d}_k)$$

$$\text{Suppose } |f_y| \leq L, |f_t + f_y f| \leq D \text{ for constants } L, D$$

$$y(t_{k+1}) - y_{k+1} = y(t_k) - y_k + h(f(t_k, y(t_k)) - f(t_k, y_k)) + \frac{h^2}{2}(f_t + f_y f)(\mathbb{c}_k, y(\mathbb{c}_k))$$

$$\Rightarrow |t_{k+1} - y_{k+1}| = E_{k+1} = \left| y(t_k) - y_k + h(y(t_k) - y_k) f_y(t_k, \mathbb{d}_k) + \frac{h^2}{2}(f_t + f_y f)(\mathbb{c}_k, y(\mathbb{c}_k)) \right| \leq E_k + hE_k L + \frac{h^2}{2}D$$

$$E_{k+1} \leq \underbrace{(1 + hL)}_{A} E_k + \underbrace{\frac{h^2 D}{2}}_{B}; E_0 = 0$$

$$E_n \leq AE_{n-1} + B \leq A(AE_{n-2} + B) + B \leq A^2 E_{n-2} + 2B \leq A^k E_{n-k} + \sum_{i=0}^{k-1} A^i B$$

$$\text{Let } k = n$$

$$= \frac{B - A^n B}{1 - A} = \frac{1 - (1 + hL)^n}{1 - 1 + h}\frac{h^2 D}{2} \Rightarrow E_n \leq \frac{((1 + hL)^n - 1)hD}{2L}$$

$$E_n \leq \frac{\left(e^{(hL)^n} - 1\right)Dh}{2L} = \underbrace{\frac{(e^{aL} - 1)D}{2L}}_{\text{constant with respect to step size}} h$$

$$\therefore \text{We say the error is } O(h) \qquad \text{(asymptotic towards 0)}$$

$$\text{or } O\left(\frac{1}{n}\right) \qquad \text{(asymptotic towards } \infty)$$

Example

Suppose the error is estimating the value of a solution to a 1st order IVP using Euler's method is approximated to be no more than 0.1. What changes should we make in order to guarantee the error does not exceed 0.01?

Solution

Decrease the step size by a factor of 10 because the error is $O(h)$. (multiply h by $\frac{1}{10}$ or multiply n by 10).

Alternative Approximations

$$y(t+h) = y(t) + \int_t^{t+h} y'(s)\mathrm{d}s = y(t) + \int_t^{t+h} f(s, y(s))\mathrm{d}s$$

Left Endpoint

$$\begin{gathered}\approx y(t) + hf(t, y(t)) \\ \Rightarrow y(t_{k+1}) \approx y(t_k) + hf(t_k, y(t_k)) \\ y_{k+1} = y_k + hf(t_k, y_k), \text{which is the Euler's Method}\end{gathered}$$

Runge-Midpoint

$$\begin{gathered}\int_t^{t+h} f(s, y(s))\mathrm{d}s \approx \underbrace{h}_{\text{base}} \cdot f\left(\underbrace{t + \frac{h}{2}}_{\text{midpoint}}, y\left(t + \frac{h}{2}\right)\right) \\ \Rightarrow y_{k+1} = y_k + hf\left(t_k + \frac{h}{2}, y_{k+\frac{1}{2}}\right) \\ f_k = f(t_k, y_k); y_{k+\frac{1}{2}} = y_k + \frac{h}{2}f_k \\ E_n = O(h^2) \qquad \text{(asymptotic towards 0)} \\ f_k = f(t_k, y_k), t_{k+\frac{1}{2}} = t_k + \frac{h}{2} \\ y_{k+\frac{1}{2}} = y_k + \frac{h}{2}f_k, f_{k+\frac{1}{2}} = f\left(t_{k+\frac{1}{2}}, y_{k+\frac{1}{2}}\right) \\ y_{k+1} = y_k + hf_{k+\frac{1}{2}}\end{gathered}$$

Runge-Trapezoidal

$$\begin{gathered}f_k = f(t_k, y_k), \widetilde{y}_{k+1} = y_k + hf_k \\ \widetilde{f}_{k+1} = f\left(t_{k+1}, \widetilde{f}_{k+1}\right) \\ y_{k+1} = y_k + \frac{h}{2}\left(f_k + \widetilde{f}_{k+1}\right) \\ E_n = O(h^2) \qquad \text{(asymptotic towards 0)}\end{gathered}$$

Runge-Kutta

Apply Simpson’s approximation to $\int_t^{t+h} f(s, y(s))\mathrm{d}s$

Error is $O(h^4) \qquad \text{(asymptotic towards 0)}$

Simpson's rule: for every three points, make a parabola out of them, then integrate that approximation rather than the original graph.

Example

Approximate $y(0.2)$ using midpoint and trapezoidal methods where y satisfies $\frac{\mathrm{d}y}{\mathrm{d}t} = t + y^2, y(0) = 1$. Use 1 step.

Solution

Midpoint

$$n = 1, h = 0.2, t_0 = 0, t_{\frac{1}{2}} = 0.1, t_1 = 0.2, y_0 = 1$$

$$f_0 = f(0, 1) = 0 + 1^2 = 1$$

$$y_{\frac{1}{2}} = 1 + 0.1 \cdot 1 = 1.1$$

$$f_{\frac{1}{2}} = 0.1 + (1.1)^2 = 0.1 + 1.21 = 1.31$$

$$y(0.2) \approx 1 + 1.31 \cdot 0.2 = 1.262$$

Trapezoidal

$$\tilde{y}_1 = 1 + 1 \cdot 0.2 = 1.2$$

$$\tilde{f}_1 = 0.2 + 1.2^2 = 0.2 + 1.44 = 1.64$$

$$y_1 = 1 + \frac{0.2}{2} \cdot (1 + 1.64) = 1 + 0.1 \cdot (2.64) = 1.264$$

Example

$$\text{Error} \leq 0.1, n = 10$$

$$\text{How many steps to guarantee error} \leq 10^{-5}$$

Solution

Euler: Error is:

$$O(h) = O\left(\frac{1}{n}\right) \Rightarrow \text{division by } 10^4 \Rightarrow \text{multiply } n \text{ by } 10^4$$

Midpoint and Trapezoidal: $O(h^2) \Rightarrow n = 1000$ steps

Runge-Kutta: $O(h^4) \Rightarrow 100$ steps

Chapter 7: Higher Order Linear Equations

Definition of Linear Equations

$$y^{(n)} + a_n(t)y^{(n-1)} + \cdots a_2(t)y'(t) + a_1 y(t) = f(t)$$

$$\text{Let } D = \frac{\mathrm{d}}{\mathrm{d}t}. \text{ Then,}$$

$$D^n[y] + a_n(t)D^{n-1}[y] + \cdots + a_2(t)D_1[y] + a_1 y = f(t)$$

$$\Rightarrow L = D^n + a_n D^{(n-1)} + \cdots + a_2(t)D + a_1(t)I$$

$$L(y) = f(t)$$

$$L[\mathbb{c}_1 y_1 + \mathbb{c}_2 y_2] = \mathbb{c}_1 L[y_1] + \mathbb{c}_2 L[y_2]$$

$$\text{for constants } \mathbb{c}_1, \mathbb{c}_2 \text{ and functions } y_1, y_2$$

Uniqueness and Existence Theorem for Differential Linear Equations

Let I be an open interval, and let $a_1(t), ..., a_{n(t)}, f(t)$ be continuous functions over I. For every $t_0 \in I$ and every $y_0, ..., y_{n-1} \in \mathbb{R}$ the following initial value problem has a unique solution:

$$\begin{cases} y^{(n)} + a_n(t)y^{(n-1)} + \cdots + a_2(t)y' + a_1(t)y = f(t) \\ y(t_0) = y_0 \\ y'(t_0) = y_1 \\ \vdots \\ y^{n-1}(t_0) = y_{n-1} \end{cases}$$

General Solution to a linear differential equation from a particular solution

The general solution to a linear differential equation is $\mathrm{Ker}\ L + Y_p$, where Y_p is some solution to the equation.

General Solution

A solution involving some constants, where changing constants will yield all possible solutions.

Example

Show $y = \mathbb{c}_1 e^t + \mathbb{c}_2 e^{-t}$ with $\mathbb{c}_1, \mathbb{c}_2 \in \mathbb{R}$, is the general solution to $y'' - y = 0$

Solution

First, show that $y = \mathbb{c}_1 e^t + \mathbb{c}_2 e^{-t}$ satisfies the given differential equation.

$$\text{Let } L = D^2 - I \Rightarrow L[y] = \mathbb{c}_1 L[e^t] + \mathbb{c}_2 L[e^- t] = \mathbb{c}_1(e^t - e^t) + \mathbb{c}_2(e^{-t} - e^{-t}) = \mathbb{c}_1(0) + \mathbb{c}_2(0) = 0$$

Next, we will show that there are no other solutions.

Let y_1 be a solution to $y'' - y = 0$ where it satisfies the IVP:

$$\begin{cases} y'' - y = 0 \\ y(0) = y_1(0) \\ y'(0) = y_1'(0) \end{cases}$$

We will find some $\mathbb{c}_1, \mathbb{c}_2$ such that $y = \mathbb{c}_1 e^t + \mathbb{c}_2 e^{-t}$ also satisfies the IVP.

$$y_1(0) = \mathbb{c}_1 + \mathbb{c}_2$$
$$y_1'(0) = \mathbb{c}_1 - \mathbb{c}_2$$
$$y_1(0) + y_1'(0) = 2\mathbb{c}_1$$
$$y_1(0) - y_1'(0) = 2\mathbb{c}_2$$
$$\Rightarrow \mathbb{c}_1 = \frac{y_1(0) + y_1'(0)}{2}, \mathbb{c}_2 = \frac{y_1(0) - y_1'(0)}{2}$$

Therefore, for any y_1, where y_1 is *any* solution $y'' - y = 0$, $y = \mathbb{c}_1 e^t + \mathbb{c}_2 e^{-t}$ can be that solution. Because it can represent any solution, and all solutions satisfy the differential equation, $y = \mathbb{c}_1 e^t + \mathbb{c}_2 e^{-t}$ is the general solution to $y'' - y = 0$

Particular solution

Some solution to a differential equation.

Find the General Solution to a linear differential equation

1. Find the general solution $Y_H(t)$ to $L[y] = 0$
2. Find a particular solution $Y_p(t)$ to $L[y] = f[t]$
3. The general solution to $L[y] = f[t]$ is Y_H + Y_p

Find the general solution to homogenous linear equations

Example

Find the general solution to $y'' - 5y' + 4y = 0$

Solution

First, because the coefficients are constant, assume a solution is in the form $y = e^{\mathbb{c}t}$.

$$y = e^{\mathbb{c}t}, y' = \mathbb{c}e^{\mathbb{c}t}, y'' = \mathbb{c}^2 e^{\mathbb{c}t}$$
$$\Rightarrow \mathbb{c}^2 e^{\mathbb{c}t} - 5\mathbb{c}e^{\mathbb{c}t} + 4e^{\mathbb{c}t} = 0$$
$$\Rightarrow (\mathbb{c}^2 - 5\mathbb{c} + 4)e^{\mathbb{c}t} = 0$$
$$\Rightarrow \mathbb{c}^2 - 5\mathbb{c} + 4 = 0 \Rightarrow \mathbb{c} \in \{1, 4\}.$$
$$\Rightarrow y = e^t, e^{4t} \text{ are solutions}$$
$$\therefore \text{We claim } y = \mathbb{c}_1 e^t + \mathbb{c}_2 e^{4t} \text{ is a general solution.}$$

$$\text{Set } L = D^2 - 5D + 4I.$$
$$\text{By linearity, } L[\mathbb{c}_1 e^t + \mathbb{c}_2 e^{4t}] = 0 \text{ so } \mathbb{c}_1 e^t + \mathbb{c}_2 e^{4t} \text{ is a solution.}$$
$$\text{Let } y_1 \text{ be a solution.}$$
$$\text{To be a solution, it must fulfill the following IVP, which uniquely constrains it by the E \& U theorem.}$$
$$\begin{cases} L[y] = 0 \\ \mathbb{c}_1 + \mathbb{c}_2 = y_1(0) \\ \mathbb{c}_1 + 4\mathbb{c}_2 = y_1'(0) \end{cases}$$
$$\text{We can find } \mathbb{c}_1, \mathbb{c}_2 \text{ such that they fulfill the IVP : } \begin{cases} \mathbb{c}_1 = \frac{4y_1(0) - y_1'(0)}{3} \\ \mathbb{c}_2 = \frac{y_1'(0) + y_1(0)}{3} \end{cases}$$
$$\text{Since we can generate all solutions, our solution is a general solution.}$$

Is this always the general solution?

$$y = \mathbb{c}_1 e^{r_1 t} + \mathbb{c}_2 e^{r_2 t} + \cdots + \mathbb{c}_n e^{r_n t}$$
$$\Rightarrow \begin{cases} \mathbb{c}_1 + \mathbb{c}_2 + \ldots + c_n = y_0 \\ r_1 \mathbb{c}_1 + r_2 \mathbb{c}_2 + \ldots + r_2 c_n = y_1 \\ \vdots \\ r_1^{n-1} \mathbb{c}_1 + r_2^{n-1} \mathbb{c}_2 + \ldots + r_2^{n-1} c_n = y_1 \end{cases} \Rightarrow$$
$$\begin{pmatrix} 1 & \cdots & 1 \\ r_1 & \cdots & r_n \\ & \vdots & \\ r_1^{n-1} & \cdots & r_n^{n-1} \end{pmatrix} \begin{pmatrix} \mathbb{c}_1 \\ \vdots \\ \mathbb{c}_n \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

Dimension of the Solution Set for a Linear Equation

Suppose $L[y] = 0$ is a nth order linear equation with continuous coefficients over an interval (a, b). Then, the solution set is an n-dimensional vector space.

Proof

This solution set is $\text{Ker } L$, which is a subspace.

Fix $t_0 \in (a, b)$. Every solution $y = Y$ to $L[y] = 0$ satisfies some IVP:

$$\begin{cases} L[y] = 0 \\ y(t_0) = Y(t_0) \\ \vdots \\ y^{(n-1)}(t_0) = Y^{(n-1)}(t_0) \end{cases}$$

Let $N_j(t)$ be the solution to

$$\begin{cases} L[y] = 0 \\ y(t_0) = 0 \\ \vdots \\ y^{(j)}(t_0) = 1 \\ y^{(n-1)}(t_0) = 0 \end{cases}$$

Linear Independence:

$$\text{Suppose} \sum_{j=0}^{n-1} \mathbb{c}_j N_j(t) = 0 \forall t \in (a,b)$$

Substitute $t = t_0$

$$\sum \mathbb{c}_j N_k(t) = 0 \Rightarrow \mathbb{c}_0 N_0(t_0) + \cdots + \mathbb{c}_{n-1} N_{n-1}(t_0) = 0 \Rightarrow \mathbb{c}_0 = 0 \text{ as } N_0(t_0) = 1, \text{and } N_j(t_0) = 0 \forall n = 1, ..., n-1.$$

$$\text{Then differentiate: } \mathbb{c}_1 N_1'(t) + \cdots + \mathbb{c}_{n-1} N_{n-1}'(t) = 0 \Rightarrow \mathbb{c}_1 = 0$$

$$\vdots$$

$$\mathbb{c}_0 = \mathbb{c}_1 = \cdots = \mathbb{c}_{n-1} = 0$$

Spanning:

$$\text{Let } Y \text{ satisfy : } \begin{cases} L[y] = 0 \\ y(t_0) = y_0 \\ \vdots \\ y^{(n-1)}(t_0) = y_{n-1} \end{cases}$$

$$\text{Claim: } Y(t) = y_0 N_0(t) + y_1 N_1(t) + \cdots + y_{n-1} N_{n-1}(t)$$

$$L[Y] = L[y_0 N_0(t) + y_1 N_1(t) + \cdots + y_{n-1} N_{n-1}(t)] =$$

$$y_0 L[N_0(t)] + y_1 L[N_1(t)] + \cdots + y_{n-1} L[N_{n-1}(t)] =$$

$$y_0(0) + y_1(0) + \cdots + y_{n-1}(0) = 0 \Rightarrow L[Y] = 0$$

$$Y^{(k)}(t_0) = y_0 N_0^{(k)}(t_0) + \cdots + y_{n-1} N_{n-1}^{k}(t_0) = y_k \text{ if } k = 0, ..., n-1$$

Thus, by the existence and uniqueness theorem, the claim holds

$$\therefore \square$$

Fundamental Set of Solutions

A basis for $\mathrm{Ker}(L[y])$, where $L[y] = D^n + a_n(t) D^{n-1} + \cdots + a_2(t) D + a_1(t)$

Natural Fundamental Set of Solutions

$$\text{Let } N_j(t) \text{ be the solution to}$$

$$\begin{cases} L[y] = 0 \\ y(t_0) = 0 \\ \vdots \\ y^{(j)}(t_0) = 1 \\ y^{(n-1)}(t_0) = 0 \end{cases}$$

The set $\{N_0(t), N_1(t), ..., N_{n-1}(t)\}$ is called the natural fundamental set of solutions.

Example 7.4

Given that e^t, e^2t are solutions to $y'' - 3y' + 2y = 0$, find the NFSOS at $t = 0$

Solution

$$W[e^t, e^{2t}] = \det\begin{pmatrix} e^t & e^{2t} \\ e^t & 2e^{2t} \end{pmatrix} = e^{4t} \neq 0$$

$$\Rightarrow y = \mathbb{c}_1 e^t + \mathbb{c}_2 e^{2t} \text{ is the general solution.}$$

$$\begin{cases} y'' - 3y' + 2y = 0 \\ y(0) = y_0 \\ y'(0) = y_1 \end{cases} \quad \text{satisfies}$$

$$y = \mathbb{c}_1 e^t \mathbb{c}_2 e^{2t} \Rightarrow \begin{cases} \mathbb{c}_1 + \mathbb{c}_2 = y_0 \\ \mathbb{c}_1 + 2\mathbb{c}_2 = y_1 \end{cases} \Rightarrow y = (2y_0 - y_1)e^t + (y_1 - y_0)e^2t$$

$$= y_0(2e^t - e^{2t}) + y_1(e^{2t} - e^t)$$

$$N_0 = 2e^t - e^{2t}, N_1 = e^{2t} - e^t \text{ forms the NFSOS at } t = 0.$$

Wronskian

Suppose $Y_1, ..., Y_n$ are solutions to $L[y] = 0$. When do these solutions form a fundamental set of solutions?

$y_1 = \mathbb{c}_1 Y_1 + \dots + \mathbb{c}_n Y_n$ has to be the general solution to $L[y] = 0$. In other words, for every $y_0, ..., y_{n-1}$, there must exist $\mathbb{c}_1, ..., \mathbb{c}_n$ such that

$$\begin{cases} \mathbb{c}_1 Y_1(t_0) + \cdots + \mathbb{c}_n Y_n(t_0) = y_0 \\ \mathbb{c}_1 Y_1'(t_0) + \cdots + \mathbb{c}_n Y_n'(t_0) = y_1 \\ \vdots \\ \mathbb{c}_1 Y_1^{n-1}(t_0) + \cdots + \mathbb{c}_n Y_n^{n-1}(t_0) = y_{n-1} \end{cases}$$

$$\Rightarrow \begin{pmatrix} Y_1(t_0) & \cdots & Y_n(t_0) \\ Y_1'(t_0) & \cdots & Y_n'(t_0) \\ & \vdots & \\ Y_1^{(n-1)}(t_0) & \cdots & Y_n^{(n-1)}(t_0) \end{pmatrix} \begin{pmatrix} \mathbb{c}_1 \\ \mathbb{c}_2 \\ \vdots \\ \mathbb{c}_n \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{pmatrix}$$

$$\therefore \det \begin{pmatrix} Y_1(t_0) & \cdots & Y_n(t_0) \\ Y_1'(t_0) & \cdots & Y_n'(t_0) \\ & \vdots & \\ Y_1^{(n-1)}(t_0) & \cdots & Y_n^{(n-1)}(t_0) \end{pmatrix} \neq 0 \iff Y_1, ..., Y_n \text{ form a fundamental set of solutions.}$$

$$\text{The determinant : } W[Y_1, ..., Y_n](t) = \det \begin{pmatrix} Y_1(t) & \cdots & Y_n(t) \\ Y_1'(t) & \cdots & Y_n'(t) \\ & \vdots & \\ Y_1^{(n-1)}(t) & \cdots & Y_n^{(n-1)}(t) \end{pmatrix} \text{ is called the Wronskian}$$

$$Y_1, ..., Y_n \text{ is a FSOS iff } \exists t_0, W[Y_1, ..., Y_n](t_0) \neq 0 \text{ iff } \forall t, W[Y_1, ..., Y_n](t) \neq 0$$

Generic Example

We know $t^2 - 1$ and t are solutions to $(1+t^2)y'' - 2ty' + 2y = 0$. Find the general solution. Use that to find a solution satisfying $y(0) = 2, y'(0) = 3$.

Solution

$$y'' - 2\frac{t}{1+t^2}y' + \frac{2}{1+t^2}y = 0$$

$$W[t^2 - 1, t] = \det\begin{pmatrix} t^2 - 1 & t \\ 2t & 1 \end{pmatrix} = t^2 - 1 - 2t^2 = -1 - t^2 \neq 0$$

$$\Rightarrow \text{the general solution is } \boxed{y = \mathbb{c}_1(t^2 - 1) + \mathbb{c}_2 t}$$

$$\begin{cases} y(0) = 2 \\ y'(0) = 3 \end{cases} \Rightarrow \begin{cases} -\mathbb{c}_1 = 2 \\ \mathbb{c}_2 = 3 \end{cases} = \boxed{y(t) = -2(t^2 - 1) + 3t}$$

Abel's Theorem

Suppose W is the Wronskian of n solutions of to an nth order linear homogenous differential equation $L[y] = 0$. Then, $\frac{\mathrm{d}W}{\mathrm{d}t} + a_n(t)W = 0$ where $L = D^n + a_n(t)D^{n-1} + \cdots + a_2(t)D + a_1(t)I$. Furthermore if $\exists t_0, W(t_0) = 0$, then $\forall t, W(t) = 0$.

Proof of the furthermore

IVP

$$\begin{cases} \frac{\mathrm{d}z}{\mathrm{d}t} + a_n(t)z = 0 \\ z(t_0) = 0 \end{cases}$$

Then, this IVP has two solutions, $z = 0, z = W \therefore \forall t, W(t) = 0$

Proof of the rest of the theorem

$$W(t) = \det \begin{pmatrix} Y_1(t) & \cdots & Y_n(t) \\ Y_1'(t) & \cdots & Y_n'(t) \\ & \vdots & \\ Y_1^{(n-1)}(t) & \cdots & Y_n^{(n-1)}(t) \end{pmatrix}$$

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \det \begin{pmatrix} Y_1'(t) & \cdots & Y_n'(t) \\ Y_1'(t) & \cdots & Y_n'(t) \\ & \vdots & \\ Y_1^{(n-1)}(t) & \cdots & Y_n^{(n-1)}(t) \end{pmatrix} + \det \begin{pmatrix} Y_1(t) & \cdots & Y_{n'}(t) \\ Y_1''(t) & \cdots & Y_n''(t) \\ & \vdots & \\ Y_1^{(n-1)}(t) & \cdots & Y_n^{(n-1)}(t) \end{pmatrix} + \cdots + \det \begin{pmatrix} Y_1(t) & \cdots & Y_{n'}(t) \\ Y_1''(t) & \cdots & Y_n''(t) \\ & \vdots & \\ Y_1^{(n)}(t) & \cdots & Y_n^{(n)}(t) \end{pmatrix}$$

$$= \det \begin{pmatrix} Y_1(t) & \cdots & Y_{n'}(t) \\ Y_1''(t) & \cdots & Y_n''(t) \\ & \vdots & \\ Y_1^{(n)}(t) & \cdots & Y_n^{(n)}(t) \end{pmatrix}$$

$$\text{Each } Y_j \text{ satisfies this differential equation:}$$

$$L[y] = y^{(n)}(t) + a_n(t)y^{(n-1)}(t) + \cdots + a_1(t)y(t) = 0$$

$$Y_j^{(n)} + a_n(t)Y_j^{(n-1)} + \cdots + a_1(t)Y_j(t) = 0$$

$$\text{Perform the following row additions:}$$

$$R_n + a_1(t)R_1 + a_2(t)R_2 + \cdots + a_{n-1}(t)R_{n-1} \to R_n$$

$$\text{This should not change the determinant because row addition doesn't change determinants.}$$

$$\text{The } j\text{th entry of the last row becomes}$$

$$Y_j^{(n)} + a_1(t)Y_j + a_2(t)Y_j' + \ldots + a_{n-1}(t)Y_j^{(n-2)} = L\left[Y_j\right] - a_nY_j^{(n-1)} = -a_n(t)Y_j^{(n-1)}$$

$$\therefore$$

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \det \begin{pmatrix} Y_1(t) & \cdots & Y_n(t) \\ Y_1'(t) & \cdots & Y_n'(t) \\ & \vdots & \\ -a_n(t)Y_1^{(n-1)} & \cdots & -a_n(t)Y_n^{(n-1)} \end{pmatrix} = -a_n(t)\det \begin{pmatrix} Y_1(t) & \cdots & Y_n(t) \\ Y_1'(t) & \cdots & Y_n'(t) \\ & \vdots & \\ Y_1^{(n-1)} & \cdots & Y_n^{(n-1)} \end{pmatrix} = -a_n(t)W$$

$$\therefore \frac{\mathrm{d}W}{\mathrm{d}t} + a_n(t)W = 0$$

Example

Suppose the Wronskian W of 3 solutions to the equation $y''' - 2ty'' - y = 0$ satisfies $W(0) = 1$. Find $W(t)$

Solution

$$y''' - 2ty'' - y = 0 => a_3(t) = -2t, a_2(t) = 0, a_1(t) = -1$$

$$\text{By Abel's theorem } \frac{\mathrm{d}W}{\mathrm{d}t} - 2tW = 0$$

$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\left[e^{-t^2}W\right] = 0$$

$$e^{-t^2}W = \mathbb{c}_1$$

$$W = \mathbb{c}_1 e^{t^2}$$

$$W(0) = 1 \Rightarrow \mathbb{c}_1 = 1 \Rightarrow W = e^{t^2}$$

Example

Prove that if the Wronskian of one FSOS to $L[y] = 0$ is constant, then the Wronskian of every FSOS is constant.

Solution

$$\text{If } W_1, W_2 \text{ are Wronskian corresponding to two FSOS}$$

$$\text{and } W_1 = \mathbb{c}_1 \text{ then } \frac{\mathrm{d}W_1}{\mathrm{d}t} + a_n(t)W_1 = 0 \Rightarrow 0 + a_n(t)\mathbb{c}_1 = 0$$

$$\Rightarrow a_n(t) = 0 \text{ because } \mathbb{c}_1 \text{ is nonzero because the Wronskian is of an FSOS.}$$

$$\Rightarrow \frac{\mathrm{d}W_2}{\mathrm{d}t} + 0W_2 = 0 \Rightarrow \frac{\mathrm{d}W_2}{\mathrm{d}t} = 0 \Rightarrow W_2 = \mathbb{c}_2.\square$$

Theorem on the Wroskian's functioning

Suppose $f_1, ..., f_n : (a, b) \to \mathbb{R}$ are $(n-1)$ times differentiable. Assume $f_1, ..., f_n$ are linearly dependent over (a, b). Then, $W[f_1, ..., f_n](t) = 0$ for all $t \in (a, b)$.

Proof

$$\exists \mathbb{c}_1, ..., \mathbb{c}_n \in \mathbb{R} \text{ not all zero such that } \forall t \in (a, b) \mathbb{c}_1 f_1(t) + \cdots + \mathbb{c}_n f_n(t) = 0$$

$$\text{Differentiate } (n-1) \text{ times}$$

$$\mathbb{c}_1 f_1'(t) + \cdots + \mathbb{c}_n f_n'(t) = 0$$

$$\vdots \mathbb{c}_1 f_1^{(n-1)}(t) + \cdots + \mathbb{c}_n f_n^{(n-1)}(t) = 0$$

$$\Rightarrow \begin{pmatrix} f_1(t) & \dots & f_n(t) \\ f_1'(t) & \dots & f_n'(t) \\ & \vdots & \\ f_1^{(n-1)}(t) & \dots & f_n^{(n-1)}(t) \end{pmatrix} \begin{pmatrix} \mathbb{c}_1 \\ \vdots \\ \mathbb{c}_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} f_1(t) & \dots & f_n(t) \\ f_1'(t) & \dots & f_n'(t) \\ & \vdots & \\ f_1^{(n-1)}(t) & \dots & f_n^{(n-1)}(t) \end{pmatrix} = 0 \Rightarrow W[f_1, ..., f_n](t) = 0$$

$\square$

Chapter 8: Linear Equations with Constant Coefficients

$$L = D^n + a_n D^{n-1} + \cdots + a_2 D + a_1 I$$
$$a_1, ..., a_n \in \mathbb{R} \text{ (constant)}$$

Example

Find the general solution to $y'' + 7y' + 12y = 0$.

Solution

$$\text{Assume } y = e^{\lambda t} \Rightarrow y' = \lambda e^{\lambda t} \Rightarrow y'' = \lambda^2 e^{\lambda t}$$
$$\Rightarrow \lambda^2 e^{\lambda t} + 7\lambda e^{\lambda t} + 12 e^{\lambda t} = 0 \Rightarrow \lambda^2 + 7\lambda + 12 = 0 \Rightarrow (\lambda+3)(\lambda+4) = 0$$
$$\lambda = -3, -4 \Rightarrow e^{-3t} \text{ and } e^{-4t} \text{ are solutions}$$
$$W[e^{-3t}, e^{-4t}] = \det\begin{pmatrix} e^{-3t} & e^{-4t} \\ -3e^{-3t} & -4e^{-4t} \end{pmatrix} = -e^{-7t} \neq 0$$
$$\therefore \boxed{y = \mathbb{c}_1 e^{-3t} + \mathbb{c}_2 e^{-4t}}$$

Example

Find the general solution to:

$$y''' + 2y'' - y' - 2y = 0$$

Solution

Let $y = e^{zt}$.

$$y' = ze^{zt}$$
$$y' = z^2 = e^{zt}$$
$$y'' = z^3 e^{zt}$$
$$\Rightarrow z^3 + 2z^2 - z - 2 = 0 \Rightarrow (z-1)(z+1)(z+2) = 0$$
$$z = 1, -1, -2 \Rightarrow y = \{e^t, e^{-t}, e^{-2t}\} \text{ are solutions}$$
$$W[e^t, e^{-t}, e^{-2t}] = \det\begin{pmatrix} e^t & e^{-t} & e^{-2t} \\ e^t & -e^{-t} & -2e^{-2t} \\ e^t & e^{-t} & 4e^{-2t} \end{pmatrix} = e^t \cdot e^{-t} \cdot e^{-2t} \det\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -2 \\ 1 & 1 & 4 \end{pmatrix}$$
$$= e^{-2t} \det\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -2 \\ 1 & (-1)^2 & (-2)^2 \end{pmatrix} \overset{\text{Vantermonte Determinant}}{=} e^{-2t} ... \therefore \neq 0$$
$$\text{Then, by a theorem, } y = \mathbb{c}_1 e^t + \mathbb{c}_2 e^{-t} + \mathbb{c}_3 e^{-2t}$$

Definition of Characteristic Polynomial

$$p(z) = z^n + a_n z^{n-1} + \cdots + a_2 z + a_1 \text{ is called the characteristic polynomial}$$
$$L = D^n + a_n D^{n-1} + \cdots + a_2 D + a_1 I$$
$$a_1, ..., a_n \in \mathbb{R} \text{ (constant)}$$

Linear Independence of Exponentials

Given distinct complex numbers $z_1, ..., z_n$, the functions $e^{z_1 t}, ..., e^{z_n t}$ are linearly independent.

Proof

If $W[e^{z_1 t}, ..., e^{z_n t}] \neq 0$ for some $t \in \mathbb{R}$ then $e^{z_1 t}, ..., e^{z_n t}$ are linearly independent.

$$W[e^{z_1 t}, \cdots, e^{z_n t}] = \det \begin{pmatrix} e^{z_1 t} & \cdots & e^{z_n t} \\ z_1 e^{z_1 t} & \cdots & z_n e^{z_n t} \\ & \vdots & \\ z_1^{n-1} e^{z_1 t} & \cdots & z_n^{n-1} e^{z_n t} \end{pmatrix}$$

$$= e^{z_1 + \cdots + z^n} t \det \begin{pmatrix} 1 & z_1 & \cdots & z_1^{n-1} \\ 1 & z_2 & \cdots & z_2^{n-1} \\ & & \vdots & \\ 1 & z_n & \cdots & z_n^{n-1} \end{pmatrix} = e^{z_1 + \cdots + z^n} t \prod_{n \geq j \geq k \geq 1} (z_j - z_k) \neq 0 \text{ because } z_j \neq z_k$$

Proof 2, electric boogalo

By induction on n

Basis step:

⋮

Inductive Step:

$$\text{Assume } z_1, ..., z_{n+1} \text{ are distinct.}$$

$$\text{Suppose} \sum_{j=1}^{n+1} \mathbb{c}_j e^{z_j t} = 0 \text{ for some } \mathbb{c}_1, ..., \mathbb{c}_{n+1} \in \mathbb{C}$$

$$\text{By differentiation } \mathbb{c}_1 z_1 e^{z_1 t} + \cdots + \mathbb{c}_n z_n e^{z_n t} + \mathbb{c}_{n+1} z_{n+1} e^{z_{n+1} t}$$

$$\sum_{j=1}^{n+1} \mathbb{c}_j e^{z_j t} - (\mathbb{c}_1 z_1 e^{z_1 t} + \cdots + \mathbb{c}_n z_n e^{z_n t} + \mathbb{c}_{n+1} z_{n+1} e^{z_{n+1} t}) = 0 - 0$$

$$\mathbb{c}_1 (z_{n+1} - z_1) e^{z_1 t} + \cdots + \mathbb{c}_n (z_{n+1} - z_n) e^{z_n t} + \mathbb{c}_{n+1} (z_{n+1} - z_{n+1}) e^{z_{n+1} t} = 0$$

$$\mathbb{c}_1 (z_{n+1} - z_1) e^{z_1 t} + \cdots + \mathbb{c}_n (z_{n+1} - z_n) e^{z_n t} = 0$$

$$\text{By the inductive hypothesis,}$$

$$\mathbb{c}_1 (z_{n+1} - z_1) = \cdots = \mathbb{c}_n (z_{n+1} - z_n) = 0$$

$$\text{Since } \forall j, z_{n+1} \neq z_j \Rightarrow \mathbb{c}_1 = \cdots = \mathbb{c}_n = 0$$

$$\therefore \sum_{j=1}^{n+1} \mathbb{c}_j e^{z_j t} = \mathbb{c}_{n+1} e^{z_{n+1} t} = 0$$

$$\Rightarrow \mathbb{c}_{n+1} = 0 \square$$

Example

Solve the equation $y'' + 2y' + 2y = 0$

Solution

$$L = D^2 + 2D + 2I$$
$$z^2 + 2z + 2 \Rightarrow (z+1)^2 + 1 = 0 \to -1 \pm i$$
$$y = e^{(-1-i)t}, e^{(-1+i)t} \text{ are solutions}$$
$$L\left[e^{(-1+i)t}\right] = 0$$
$$L\left[e^{-t}(\cos(t) + i\sin(t))\right] = 0$$
$$L\left[e^{-t}\cos(t)\right] + L\left[e^{-t} i\sin(t))\right] = 0$$
$$L\left[e^{-t}\cos(t)\right] + iL[\sin(t))] = 0$$
$$L\left[e^{-t}\cos(t)\right] + iL\left[e^{-t}\sin(t))\right] = 0$$
$$L\left[e^{-t}\cos(t)\right] = 0$$
$$L\left[e^{-t}\sin(t)\right] = 0$$

Since $e^{-t}\cos(t)$ and $e^{-t}\sin(t)$ are not scalar multiples, they are linearly independent and the general solution is:

$$\boxed{y = c_1 e^{-t}\cos(t) + c_2 e^{-t}\sin(t).}$$

In general if $a \pm b$ is a root of the characteristic polynomial, $a, b \in \mathbb{R}$ then

$$L\left[e^{at}\cos(bt)\right] = 0$$
$$L\left[e^{at}\sin(bt)\right] = 0$$

Example

Solve $y''' + 2y'' + y' = 0$

Solution

$$\Rightarrow z^3 + 2z^2 + z = 0$$
$$\Rightarrow z(z^2 + 2z + 1) = 0$$
$$\Rightarrow z(z+1)^2 = 0$$
$$\Rightarrow z = 0, -1, -1$$
$$e^{0t}, e^{-t}, e^{-t} \text{ are solutions}$$

but this is not enough, because e^{-t} is linearly dependent with e^{-t} $\Rightarrow$ we need one more solution

$$p(z) = (z+1)^2 + 2z(z+1)$$
$$P(-1) = 0, p'(-1) = 0$$
$$\left(D = \frac{\mathrm{d}}{\mathrm{d}t}\right)$$
$$(D^3 + 2D^2 + D)[e^{zt}] = (z^3 + 2z^2 + z)e^{zt}$$

We will differentiate both sides with respect to z:

$$\frac{\partial}{\partial z} D = D \frac{\partial}{\partial z} \text{ by Clairaut's Theorem}$$
$$\frac{\partial}{\partial z}(D^3 + 2D^2 + D)[e^{zt}] = \frac{\partial}{\partial z}((z^3 + 2z^2 + z)e^{zt})$$
$$(D^3 + 2D^2 + D)[te^{zt}] = (3z^2 + 4z + 1)e^{zt} + (z^3 + 2z^2 + z)te^{zt}$$
$$z = -1 \Rightarrow$$
$$(D^3 + 2D^2 + D)[te^{-t}] = 0$$
$$\Rightarrow y = 1, e^{-t}, te^{-t}$$

Apply the Wronskian or apply the definition of linear independence.

$$\vdots$$

$$y = 1, e^{-t}, te^{-t} \text{ are } \underline{\text{linearly independent}} \Rightarrow$$
$$y = \mathbb{c}_1 + \mathbb{c}_2 e^{-t} + \mathbb{c}_3 te^{-t}$$

Key Identities (Showing why n repeated roots)

$$P(D)[t^n e^{zt}] = \sum_{k=0}^{n} \binom{n}{k} p^{(n-k)}(z) t^k e^{zt}.$$

Proofish

$$\text{Suppose } L(y) = D^n + a_n D^{n-1} + \cdots + a_2 D + a_1 I = p(D), a_j \in \mathbb{R}$$

$$\forall z \in \mathbb{C}, t \in \mathbb{R}, L[e^{zt}] = p(z)e^{zt}$$

$$\text{Since } D\frac{\partial}{\partial z} = \frac{\partial}{\partial z}D$$

$$P(D)[e^{zt}] = p(z)e^{zt}$$

$$\text{take the derivative with respect to z:}$$

$$P(D)\frac{\partial}{\partial z}[e^{zt}] = \frac{\partial}{\partial z}p(z)e^{zt}$$

$$P(D)[te^{zt}] = p'(z)e^{zt} + p(z)te^{zt}$$

$$\text{take the derivative with respect to z:}$$

$$P(D)[t^2e^{zt}] = p''(z)e^{zt} + p'(z)te^{zt} + p'(z)te^{zt} + p(z)t^2e^{zt}$$

$$P(D)[t^2e^{zt}] = p''(z)e^{zt} + 2p'(z)te^{zt} + p(z)t^2e^{zt}$$

$$P(D)[t^3e^{zt}] = p'''(z)e^{zt} + 3p''(z)te^{zt} + 3p'(z)t^2e^{zt} + p(z)t^3e^{zt}$$

$$\text{Via induction on } n \text{ can prove:}$$

$$P(D)[t^ne^{zt}] = \sum_{k=0}^{n}\binom{n}{k}p^{(n-k)}(z)t^ke^{zt}.$$

Definition of multiplicity

We say $c \in C$ is a root of a polynomial $p(z)$ with multiplicity m if $p(z) = (z-c)^m q(z)$, where $q(z)$ is a polynomial and $q(c) \neq 0$.

Theorem on multiplicity after derivatives

Suppose c is a root of the polynomial $p(z)$. Then c has multiplicity m iff $p(c) = p'(c) = \cdots = p^{(m-1)}(c) = 0$ and $p^{(m)}(c) \neq 0$.

Proofish

$$\text{Start with } p(z) = (z-c)^m q(z)$$

$$p'(z) = m(z-c)^{m-1}q(z) + (z-c)^m q'(z)$$

$$= (z-c)^{m-1}\underbrace{(mq(z) + (z-c)q'(z))}_{\text{not zero at } c \text{, just another q}}$$

$$\vdots$$

$$(p-z)^0 \Rightarrow 1 \text{ so no longer works}$$

Zero derivatives from repeated roots

Suppose c is a root of multiplicity m of $p(z)$ then $P(D)[e^{ct}] = P(D)[e^{ct}] = P(D)[te^{ct}] = \cdots = P(D)[t^{m-1}e^{ct}] = 0$.

Theorem to find solutions of differential equation

Real roots r with multiplicity $m \to e^{rt}, te^{rt}, ..., t^{m-1}e^{rt}$, Non-real roots $a + bi$ with multiplicity $m \to$ $e^{at}\cos(bt), e^{at}\sin(bt), ..., t^{m-1}e^{at}\cos(bt), t^{m-1}e^{at}\sin(bt)$

$L = P(D), p(z) = 0$

1. Find all roots of $p(z)$
2. For every real root r with multiplicity m consider $e^{rt}, te^{rt}, ..., t^{m-1}e^{rt}$
3. For every pair of non-real roots $a \pm bi$ with multiplicity m consider $e^{at}\cos(bt), e^{at}\sin(bt), ..., t^{m-1}e^{at}\cos(bt), t^{m-1}e^{at}\sin(bt)$
4. Take a linear combination out of those considered for your solution.

Example

Solve each of these equations:

1. $y'''' + 6y'' + 9y = 0$
2. $\left(D^2 + I\right)^2 (D - I)^3 Dy = 0$

Solution

Equation 1

$$p(z) = z^4 + 6z^2 + 9 = \left(z^2 + 3\right)^2 \Rightarrow z = \pm\sqrt{3}i \text{ with multiplicity } 2$$
$$\text{by the theorem, } y = \mathbb{c}_1 \cos\left(\sqrt{3}t\right) + \mathbb{c}_2 \sin\left(\sqrt{3}t\right) + \mathbb{c}_3 t\cos\left(\sqrt{3}t\right) + \mathbb{c}_4 t\sin\left(\sqrt{3}t\right)$$

Equation 2

$$p(z) = \left(z^2 + 1\right)^2 (z - 1)^3 z$$

$$\begin{aligned} z &= \pm i \text{ multiplicity } 2 \\ &= 1 \text{ multiplicity } 3 \\ &= 0 \text{ multiplicity } 0 \end{aligned}$$

$$y = \mathbb{c}_1 \cos(t) + \mathbb{c}_2 \sin(t) + \mathbb{c}_3 t\cos(t) + \mathbb{c}_4 t\sin(t) + \mathbb{c}_5 e^t + \mathbb{c}_6 te^t + \mathbb{c}_7 t^2 e^t + \mathbb{c}_8$$

Reduction of Order

Suppose a linear equation $L(t)[y] = 0$ has a solution y_0. Then, the reduction of order may be used to reduce the order of this equation by setting $y(t) = y_0(t)v(t)$

Example

We know e^t to $y'' - 2y' + y = 0$

Solution

We will find the second solution by the method of reduction of order.

$$\begin{aligned} y_0 = e^t \Rightarrow y = e^t v(t) \Rightarrow y' = e^t v(t) + e^t + v'(t) \Rightarrow y'' = e^t v(t) + 2e^t v'(t) + e^t v''(t) \\ \Rightarrow e^t v(t) + 2e^t v'(t) + e^t v''(t) - 2\left(e^t v(t) + e^t + v'(t)\right) + e^t v(t) \\ = e^t v'' = 0 \Rightarrow v'' = 0 \Rightarrow v = t \text{ is a solution} \\ \Rightarrow te^t \text{ is also a solution} \end{aligned}$$

Example

Given $y = t$ is a solution to $t^2y'' - t(t+2)y' + (t+2)y = 0$ with $t > 0$, find the general solution.

Solution

We will use the method of reduction of order.

$$y_0 = t$$
$$y = tv(t) \Rightarrow$$
$$y' = v(t) + tv'(t)$$
$$y'' = 2v'(t) + v''(t) + tv(t)$$
$$v''t^3 + v'\left(2t^2 + t(-t(t+2))\right) + v(-t(t+2) + t(t+2)) = 0$$
$$\Rightarrow v''t^3 + v'\left(-t^3\right) = 0 \Rightarrow v'' = v' \Rightarrow e^t \text{ is a solution} \Rightarrow y = c_1 t + c_2 te^t$$

Method of Undetermined Coefficients

We need to find a particular solution Y_p such that $L\left[Y_p\right] = f(t)$.

Suppose the forcing is $f(t) = g(t)e^{at}\cos(bt) + h(t)e^{at}\sin(bt)$ where g and h are polynomials. Let d be the largest degree of g and h. Let $L = p(D)$ and m be the multiplicity of $a + bi$ as a root of $p(z)$. (i.e. $p^{(k)}(a+bi) = 0 \text{ for } k = 0, ..., m-1$. If $p(a+bi) \neq 0, m = 0$.) We write the Key Identities starting from the mth derivative.

$$\text{Let } z = a + bi$$
$$L\left[t^m e^{zt}\right] = p^{(m)}(z)e^{zt} + \cancel{\binom{m}{1}p^{(m-1)}(z)e^{zt}} + \cdots$$
$$L\left[t^{m+1}e^{zt}\right] = p^{(m+1)}(z)e^{zt} + \binom{m+1}{1}p^{(m)}(z)te^{zt} + \cancel{\binom{m+1}{2}p^{(m-1)}(z)e^{zt}} + \cdots$$
$$\vdots$$
$$L\left[t^{m+d}e^{zt}\right] = p^{(m+d)}(z)e^{zt} + \cdots + \binom{m+d}{d}p^{(m)}(z)t^d e^{zt} + \cancel{\binom{m}{1}p^{(m-1)}(z)e^{zt}} + \cdots$$
$$\text{So you will gather polynomials of degrees } 0, 1, ..., d \times e^{zt}$$
$$\text{By an exercise, we can get any polynomial of degree} \leq d \times e^{zt}.$$

In sum,

$$Y_p = t^m\left(A_0 + A_1 t + \cdots + A_d t^d\right)e^{at}\cos(bt) + t^m\left(B_0 + B_1 t + \cdots + B_d t^d\right)e^{at}\sin(bt)$$

then just solve for the coefficients via Key Identities.

Example

Find a particular solution to $y'' + 2y = e^{5t}$

Solution

$$L = D^2 + I, p(z) = z^2 + 2$$
$$L[e^{5t}] = p(5)e^{5t} = (5^2 + 2)e^{5t} = 27e^{5t}$$
$$\Rightarrow Y_p = \frac{1}{27}e^{5t} Y_h = \mathbb{c}_1 \cos\left(\sqrt{2}t\right) + \mathbb{c}_2 \sin\left(\sqrt{2}t\right)$$
$$\therefore y = \frac{1}{27}e^{5t} + \mathbb{c}_1 \cos\left(\sqrt{2}t\right) + \mathbb{c}_2 \sin\left(\sqrt{2}t\right)$$

Example

Find a particular solution for $y'' - 6y' + 9y = 4e^{3t}$

Solution

$$p(z) = z^2 - 6z + 9 = (z-3)^2, L = p(D)$$
$$\text{Key Identities:}$$
$$L[e^{3t}] = p(3)e^{3t} = 0 \text{ not helpful}$$
$$L[te^{3t}] = p(3)te^{3t} + p'(3)e^{3t} = 0 \text{ not helpful}$$
$$L[t^2e^{3t}] = p(3)t^2e^{3t} + 2p'(3)te^{3t} + p''(3)e^{3t} = 2e^{3t}$$
$$\Rightarrow Y_p = 2t^2e^{3t}$$

Example

Find a particular solution for $y'' + 2y' + 10y = \cos(2t)$

Solution

$p(z) = z^2 + 2z + 10, L = p(D)$

Since the forcing is $\cos(2t) = \text{Re}(e^{2it})$, we sub $z = 2i$ n the Key Identities.

$$\text{Key Identities:}$$
$$L[e^{2it}] = p(2i)e^{2it} = (6 + 4i)e^{2it}$$
$$\therefore L\left[\frac{e^{2it}}{6+4i}\right] = e^{2it} \Rightarrow L\left[\text{Re}\left(\frac{e^{2it}}{6+4i}\right)\right] = \text{Re}(e^{2it})$$
$$= \cos(2t) \Rightarrow Y_p = \text{Re}\left(\frac{e^{2it}}{6+4i}\right) = \text{Re}\ \left(\frac{(6-4i)e^{2it}}{36+16}\right) = \text{Re}\left(\frac{6\cos(2t) + 4\sin(2t) + i(...)}{52}\right)$$
$$\Rightarrow Y_p = \frac{3\cos(2t) + 2\sin(2t)}{26}$$

Example

Find a particular solution for $y'' + 2y' + 10y = 4te^{2t}$

Solution

$$p(z) = z^2 + 2z + 10, L = p(D)$$
$$z = 2 \text{ from the forcing}$$
$$\text{Key Identities:}$$
$$L[e^{2t}] = p(2)e^{2t} = 18te^{2t}$$
$$L[te^{2t}] = p(2)te^{2t} + p'(2)e^{2t} = 18te^{2t} + 6e^{2t}$$
$$\therefore L[te^{2t}] - \frac{1}{3}L[e^{2t}] = 6e^{2t} + 18te^{2t} - 6e^{2t} = 18te^{2t} = 18te^{2t}$$
$$\frac{L[te^{2t}] - \frac{1}{3}L[e^{2t}]}{18} \cdot 4 = 4te^{2t}$$
$$\Rightarrow Y_p = \frac{2}{9}te^{2t} - \frac{2}{27}e^{2t}$$

Example

Find a particular solution for $y'' + y = \sin t + t$

Solution

$$p(z) = z^2 + 1, L = p(D)$$
$$y'' + y = \underbrace{\sin t}_{z=i} + \underbrace{t}_{z=0}$$
$$\text{since } z = i \text{ is a simple root of } p(z) \text{ the first Key Identity does not help.}$$
$$L[te^{it}] = \cancel{p(i)te^{it}}^{0} + p'(i)e^{it} = 2ie^{it}$$
$$L\left[\frac{te^{it}}{2i}\right] = L\left[\operatorname{Im}\left(\frac{te^{it}}{2i}\right)\right] = \sin(t) \Rightarrow L\left[\frac{-t\cos t}{2}\right] = \sin(t)$$
$$z = 0:$$
$$L[1] = p(0)e^{0t} = p(0) = 1$$
$$L[t] = p(0)te^{0t} + p'(0)e^{0t} = t$$
$$\Rightarrow L\left[\frac{-t\cos t}{2}\right] = \sin(t), L[t] = t$$
$$\Rightarrow L\left[t - \frac{t\cos t}{2}\right] = \sin(t) + t \therefore \boxed{Y_p = t - \frac{t\cos t}{2}}$$

Example using the theorem

Find a particular solution using the method of undetermined coefficients.

1. $y'' + 4t = t\cos(2t)$
2. $y' - 6y' + 9y = 4e^{3t}$
3. $y' + 2y' + 10y = 5e^{-t}\sin(3t)$
4. $y'' + 3y' - 4y = 2\sin(t)\cos(3t)$

Solution

For 1:

$$a = 0, b = 2, p(z) = z^2 + 4, d = 1$$
$$p(2i) = 0, p'(2i) = 4i \Rightarrow m = 1$$
$$Y_p = t(A_0 + A_1 t)e^{0t}\cos(2t) + t(B_0 + B_1 t)e^{0t}\sin(2t)$$
$$= A_0 t\cos(2t) + A_1 t^2\cos(2t) + B_0 t\sin(2t) + B_1 t^2\sin(2t)$$
substitute into the equation and set the coefficients equal to 0 and solve the system.

$$\vdots$$

For 2:

$$p(z) = z^2 - 6z + 9 = (z - 3), a + bi = 3, m = 2, d = 0$$
$$Y_p = t^2(A_0)e^{3t}$$
$$Y_p' = A_0 t^2 \cdot 3e^{3t} + 2A_0 t \cdot e^{3t}$$
$$Y_p'' = 6A_0 te^{3t} + 9A_0 t^2 e^{3t} + 2A_0 e^{3t} + 6A_0 te^{3t}$$
$$Y_p'' - 6Y_p' + 9Y_p = e^{3t}(2A_0) + \cdots$$
$$\Rightarrow A_0 = 2$$
$$Y_p = 2t^2 e^{3t}$$

For 3:

$$p(z) = z^2 + 2z + 10; a + bi = 1 + 3i, d = 0, m = 1$$
$$Y_p = tA_0 e^{-t}\cos(3t) + tB_0 e^{-t}\sin(3t):$$

For 4:

We will have to convert $2\sin(t)\cos(3t)$ to a sum

$$2\sin(t)\cos(3t)$$
$$= 2\frac{e^{it} - e^{-it}}{2i}\frac{e^{i3t} + e^{-i3t}}{2}$$
$$= \frac{e^{it} - e^{-it}}{2i}(e^{i3t} + e^{-i3t})$$
$$= -\frac{e^{-4it}}{2i} + \frac{e^{-2it}}{2i} - \frac{e^{2it}}{2i} + \frac{e^{4it}}{2i}$$
$$= \sin(4t) - \sin(2t)$$

Find Y_p by forcing $\sin(4t)$ and $\sin(2t)$ separately

$$\vdots$$

Variation of Parameters

$L[y] = 0$ has a known FSoS $Y_1, ..., Y_n$.

Objective: find a particular solution for $L[y] = f(t)$

$$Y_p = u_1 Y_1 + \cdots + u_n Y_n \text{ where } u_1, ..., u_n \text{ are functions of } t$$

$$Y_p' = u_1 Y_1' + \cdots + u_n Y_n' + \underbrace{u_1' Y_1 + \cdots + u_n' Y_n}_{\text{Let this be 0 hopefully}}$$

$$Y_p'' = u_1 Y_1'' + \cdots + u_n Y_n'' + \underbrace{u_1' Y_1' + \cdots + u_n' Y_n'}_{0}$$

$$\vdots$$

$$Y_p^{(n-1)} = u_1 Y_1^{(n-1)} + \cdots + u_n Y_n^{(n-1)} + \underbrace{u_1' Y_1^{(n-2)} + \cdots + u_n' Y_n^{(n-2)}}_{0}$$

$$Y_p^{(n)} = u_1 Y_1^{(n)} + \cdots + u_n Y_n^{(n)} + u_1' Y_1^{(n-1)} + \cdots + u_n' Y_n^{(n)}$$

$$L[Y_p] = u_1 \cancel{L[Y_1]} + u_2 \cancel{L[Y_2]} + \cdots + u_n \cancel{L[Y_n]} + u_1' Y_1^{(n-1)} + ... + u_n' Y_1^{(n-1)} = f(t)$$

$$\begin{cases} u_1' Y_1 + \cdots + u_n' Y_n = 0 \\ u_1' Y_1' + \cdots + u_n' Y_n' = 0 \\ \vdots \\ u_1' Y_1^{(n-2)} + \cdots + u_n' Y_n^{(n-2)} = 0 \\ u_1' Y_1^{(n-1)} + \cdots + u_n' Y_n^{(n-1)} = f(t) \end{cases}$$

$$\begin{pmatrix} Y_1 & \cdots & Y_n \\ Y_1' & \cdots & Y_n' \\ & \vdots & \\ Y_1^{(n-1)} & \cdots & Y_n^{(n-1)} \end{pmatrix} \begin{pmatrix} u_1' \\ \vdots \\ u_n' \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ f(t) \end{pmatrix}$$

$$\text{Note we only need one ntuple } (u_1, ..., u_n)$$

Example

Find a solution for $y'' + y = \tan(t)$

Solution

$$\text{Make sure the equation is in standard form } Y_1 = \cos(t), Y_2 = \sin(t)$$

$$Y_p = u_1 \cos(t) + u_2 \sin(t)$$

$$\Rightarrow \begin{cases} u_1' \cos(t) + u_2' \sin(t) = 0 \\ -u_1' \sin(t) + u_2' \cos(t) = \tan(t) \end{cases}$$

Chapter 9: Power Series Solutions

The Idea

$$y'' + p(t)y' + q(t)y = 0$$

$$\text{Write } y = \sum_{n=0}^{\infty} a_n (t - t_0)^n \qquad (*)$$

$$\text{Substitute } y, y', y'', \text{, then obtain a recursion } a_n$$

$$\text{Solve for } a_n$$

If we know a/all solution of the form $(*)$ exists, then we are done.

If can show $(*)$ converges, then $(*)$ would be a solution.

Analytic

$f(t)$ is analytic at t_0 if $f(t) = \sum_{n=0}^{\infty} a_n (t-t_0)^n \exists R > 0 \forall t \in (t_0 - R, t_0 + R)$

Composition of analytic functions via addition, multiplication, and division

Suppose $f(t)$ and $g(t)$ are analytic at $t = t_0$. Then the functions $f(t) + g(t)$ and $f(t)g(t)$ are also analytic at $t = t_0$. Furthermore, if $g(t_0) \neq 0$, then $\frac{f(t)}{g(t)}$ is analytic at t_0.

Ordinary Point

A point t_0 is said to be an ordinary point for the equation $y'' + p(t)y' + q(t)y = 0$ if $p(t)$ and $q(t)$ are analytic functions at t_0. If the equation is not in normal form but can be written in normal form in such a way that the coefficients are analytic at $t = t_0$, we still call t_0 an ordinary point of the equation.

Theorem for showing that an answer is analytic

Suppose $p(t)$ and $q(t)$ are analytic at t_0. Then every solution to the equation $y'' + p(t)y' + q(t)y = 0$ is analytic at t_0. Furthermore, the radius of convergence of the Taylor series of each solution centered at t_0 is at least the minimum of the radii of convergence of Taylor series $p(t)$ and $q(t)$ centered at t_0.

Example

Solve $y'' - ty = 0$

Solution

0 and $-t$ are analytic with radii of convergence ∞. By the previous theorem, every solution is analytic.

$$y = \sum_{n=0}^{\infty} a_n t^n$$

$$y' = \sum_{n=1}^{\infty} n a_n t^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2}$$

$$ty \Rightarrow \sum_{n=0}^{\infty} a_n t^{n+1} \overset{n=n-1}{=} \sum_{n=1}^{\infty} a_{n-1} t^n$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} \overset{n=n+2}{=} \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} t^n y'' - ty = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} t^n - \sum_{n=1}^{\infty} a_{n-1} t^n$$

$$= (0+2)(0+1) a_{0+2} t^0 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} t^n - \sum_{n=1}^{\infty} a_{n-1} t^n$$

$$= 2a_2 + \sum_{n=1}^{\infty} ((n+2)(n+1) a_{n+2} - a_{n-1}) t^n = 0$$

$$\Rightarrow a_2 = 0 \wedge (n+2)(n+1) a_{n+2} = a_{n-1} \forall n \geq 1$$

$$a_2 = 0$$

$$a_{n+2} = \frac{a_{n-1}}{(n+2)(n+1)}$$

$$a_3 = \frac{a_0}{3 \times 2}$$

$$a_4 = \frac{a_1}{4 \times 3}$$

$$a_5 = \frac{a_2}{5 \times 4} = 0$$

$$a_6 = \frac{a_3}{6 \times 5} = \frac{a_0}{6 \times 5 \times 3 \times 2}$$

$$a_7 = \frac{a_4}{7 \times 6} = \frac{a_0}{7 \times 6 \times 4 \times 3}$$

$$a_8 = \frac{a_5}{8 \times 7} = 0$$

Claim :

$$a_{3n} = \frac{a_0}{(3n)(3n-1)\ (3n-3)(3n-4)\ (3n-6)(3n-7) \cdots \times 3 \times 2}$$

$$a_{3n+1} = \frac{a_1}{(3n+1)(3n)\ (3n-2)(3n-3) \cdots \times 4 \times 3}$$

$$a_{3n+2} = 0$$

$\vdots$ prove by induction

Therefore, the general solution is

$$y = \sum_{n=0}^{\infty} (a_3 n) t^{3n} + \sum_{n=0}^{\infty} a_{3n+1} t^{3n+1}$$

Example
Solve $y'' + t^2 y' + 2ty = 0, y(0) = 1, y'(0) = 0$.

Solution
t^2 and $2t$ are analytic with a radius of convergence of ∞. Thus, the solution to this IVP (which is unique by the existence and uniqueness theorem for linear equations) is also analytic with a radius of convergence ∞.

$$y=\sum_{n=0}^{\infty}a_nt^n$$

$$y'=\sum_{n=1}^{\infty}na_nt^{n-1}$$

$$y''=\sum_{n=2}^{\infty}n(n-1)a_nt^{n-2}$$

$$y''=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}t^n$$

$$t^2y'=\sum_{n=1}^{\infty}na_nt^{n+1}=\sum_{n=2}^{\infty}(n-1)(a_{n-1})t^n$$

$$2ty=\sum_{n=0}^{\infty}2a_nt^{n+1}=\sum_{n=1}^{\infty}2a_{n-1}t^n$$

$$y''+t^2y'+2ty=$$

$$2a_2+6a_3t+2a_0t+\sum_{n=2}^{\infty}((n+2)(n+1)a_{n+2}+(n-1)a_{n-1}+2a_{n-1})t^n$$

$$\begin{cases}2a_2=0\\6a_3+2a_0=0\\(n+2)(n+1)a_{n+2}+(n+1)a_{n-1}=0\forall n\geq 2\end{cases}\Rightarrow$$

$$\begin{cases}a_2=0\\a_3=-\frac{a_0}{3}\\a_{n+2}=-\frac{a_{n-1}}{n+2}\end{cases}\Rightarrow$$

$$a_4=-\frac{a_1}{4}\overset{\text{by initial conditions}}{=}0$$

$$a_5=-\frac{a_2}{5}$$

$$a_6=-\frac{a_3}{6}=\frac{a_0}{6\times 3}$$

$$a_7=-\frac{a_4}{7}=0$$

$$a_8=-\frac{a_5}{8}=0$$

$$a_9=-\frac{a_6}{9}=-\frac{a_0}{9\times 6\times 3}$$

$$\Rightarrow a_{3n+1}=a_{3n+2}=0\forall n\geq 0$$

$$a_{3n}=\frac{(-1)^na_0}{3^nn!}=\frac{(-1)^n}{3^nn!}$$

$$y=\sum_{n=0}^{\infty}\frac{(-1)^n}{3^nn!}t^{3n}=\sum_{n=0}^{\infty}\frac{\left(-\frac{t^3}{3}\right)^n}{n!}=e^{-\frac{t^3}{3}}$$

Example
Solve

$$y'' + (t^2 + 2t + 1)y' - (4t + 4)y = 0$$
$$y(-1) = 0, y'(-1) = 1$$

Solution
Note that both $t^2 + 2t + 1$ and $-(4t + 4)$ are analytic at $t_0 = -1$ with a radius of convergence of ∞.

$$\text{Set } s = t + 1, z(s) = y(t + 1)$$
$$\begin{cases} z'' + s^2 z' - 4sz = 0 \\ z(0) = 0 \\ z'(0) = 1 \end{cases}$$
$$z = \sum_{n=0}^{\infty} a_n s^n \Rightarrow -4sz = \sum_{n=0}^{\infty} -4a_n s^{n+1} = \sum_{n=1}^{\infty} -4a_{n-1} s^n$$
$$z' = \sum_{n=1} n a_n s^{n-1} \Rightarrow s^2 z' = \sum_{n=1}^{\infty} n a_n z^{n+1} = \sum_{n=2}^{\infty} (n-1) a_{n-1} s^n$$
$$z'' = \sum_{n=1} n(n-1) a_n s^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} s^n$$
$$z'' + s^2 z' - 4sz =$$
$$2a_2 + 6a_3 s - 4a_0 s + \sum_{n=2}^{\infty} (-4a_{n-1} + (n-1)a_{n-1} + (n+2)(n+1)a_{n+2}) s^n = 0$$
$$\Rightarrow \begin{cases} a_2 = 0 \\ a_3 = 2\frac{a_0}{3} \\ a_{n+2} = -\frac{(n-5)a_{n-1}}{(n+2)(n+1)} \end{cases}$$
$$z(0) = 0 \Rightarrow a_0 = 0 \Rightarrow a_3 = 0$$
$$z'(0) = 1 \Rightarrow a_1 = 1$$
$$a_4 = -\frac{(-3)a_1}{4 \times 3} = \frac{1}{4}$$
$$a_5 = -\frac{(-2)(a_2)}{5 \times 4} = 0$$
$$a_6 = -\frac{(-1)(a_3)}{6 \times 5} = 0$$
$$a_7 = 0, a_8 = 0, a_9 = 0$$
$$\text{By induction } a_n = 0, n \geq 5, z = s + \frac{s^4}{4} \therefore \text{the solution to the IVP is}$$
$$y = t + 1 + \frac{(t+1)^4}{4}$$

Singular Point

$p(t)$ is unbounded near to t_0 or $q(t)$ is unbounded near t_0

$$\forall t \in (t_0 - \varepsilon, t_0 + \varepsilon) - \{t_0\}, \neg \exists M, |p(t)| \le M \wedge |q(t)| \le M$$

and p, q are continuous on $(t_0 - \varepsilon, t_0 + \varepsilon) - \{t_0\}$.

Example

$$\left\{p(t) = \tfrac{\sin(t)}{t} q(t) = \cos(t) \Rightarrow \frac{\sin(t)}{t} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n+1)!} \text{ is analytic}, 0 \text{ is not a singular point}\right.$$

$$\begin{cases} p(t) = \frac{\cos(t)}{t} \\ q(t) = \sin(t) \end{cases} \Rightarrow t = 0 \text{ is a singular point}$$

Series solutions near regular singular points

Euler's Equation

$t^2 y'' + \alpha t y' + \beta y = 0, \alpha, \beta \in \mathbb{R}$ is called an Euler's equation.

$r(r-1) + \alpha r + \beta = 0$ is called the indicial equation.

Solution

$$y = t^r \Rightarrow y' = r t^{r-1}, y'' = r(r-1)t^{r-2}$$

$$t^2 y'' + \alpha t y' + \beta y = (r(r-1) + \alpha r + \beta) t^r = 0$$

$$r(r-1) + \alpha r + \beta = 0 \therefore \text{solve for } r \to 2 \text{ solutions, probably}$$

Find the two roots of the equation. If the two roots are distinct and real, then you have two linearly independent solutions. If they are not distinct, you multiply an answer by $\log(t)$. If you have nonreal roots, you get stuff with $\cos$ and $\sin$

Example

Solve for $t > 0$:

1. $t^2 y'' + 6 t y' + 4y = 0$
2. $t^2 y'' + 3 t y' + y = 0$
3. $4t^2 y'' + 20 t y' + 25 y = 0$

Solution

1.

$$\text{Set } y = t^r$$
$$y' = rt^{r-1}, y'' = r(r-1)t^{r-2}$$
$$t^2y'' + 6ty' + 4y = (r(r-1) + 6r + 4)t^r = 0$$
$$\Rightarrow r(r-1) + 6r + 4 = 0$$
$$\Rightarrow r = \{-1, -4\}$$
$$\Rightarrow t^{-1} \wedge t^{-4} \text{ are two } \underline{\text{linearly independent}} \text{ functions by the fact they are not scalar multiples of each other}$$
$$\therefore y = \frac{\mathbb{c}_1}{t} + \frac{\mathbb{c}_2}{t^4}$$

2.

$$r^2 + 2r + 1 = 0 \Rightarrow r = \{-1, -1\} \Rightarrow y = t^{-1} \text{ is a solution}$$
$$\text{We will find another solution via reduction of order.}$$
$$y = t^{-1}v \Rightarrow y' = -t^{-2}v + t^{-1}v', y'' = 2t^{-3}v - 2t^{-2}v' + t^{-1}v''$$
$$t^2y'' + 3ty' + y = tv'' + (-2+3)v' + (0)v = 0$$
$$\overset{w=v'}{\Rightarrow} tw' + w = 0 \Rightarrow (tw)' = 0 \Rightarrow (tw) = 1 \Rightarrow w = \frac{1}{t} \Rightarrow v = \log(t) \text{ is a solution}$$
$$\Rightarrow y = \frac{\log(t)}{t} \text{ is a solution}$$
$$\Rightarrow y = \frac{\mathbb{c}_1}{t} + \frac{\mathbb{c}_2 \log(t)}{t} \text{ is the general solution}$$

3.

$$4t^2y'' + 20ty' + 25y = 0$$
$$\Rightarrow 4r^2 + 16r + 25 = 0 \Rightarrow (2r+4)^2 + 9 = 0 \Rightarrow r = -2 \pm \frac{3}{2}i$$
$$t^r = t^{-2}t^{\frac{3}{2}i} = t^{-2}e^{\frac{3}{2}i\log(t)}$$
$$= t^{-2}\left(\cos\left(\frac{3}{2}\log(t)\right) + i\sin\left(\frac{3}{2}\log(t)\right)\right)$$
$$\text{real and imaginary parts are both solutions}$$
$$\Rightarrow \left\{t^{-2}\cos\left(\frac{3}{2}\log(t)\right), t^{-2}\sin\left(\frac{3}{2}\log(t)\right)\right\} \subset y$$
$$\Rightarrow y = \frac{\mathbb{c}_1\cos\left(\frac{3}{2}\log(t)\right) + \mathbb{c}_2\sin\left(\frac{3}{2}\log(t)\right)}{t^2}$$

Regular Singularity

We say t_0 is a <u>regular singularity</u> for $y'' + p(t)y' + q(t)y = 0$ if t_0 is a <u>singularity</u> and $(t-t_0)p(t)$ is <u>analytic</u> and $(t-t_0)^2q(t)$ is <u>analytic</u> around t_0 but not necessarily on it.

Motivation

$$(t-t_0)^2y'' + \alpha(t-t_0)y' + \beta y = 0$$

$$p(t) = \frac{\alpha}{t-t_0}, q(t) = \frac{\beta}{(t-t_0)^2}$$

But this has a solution, so we should see why this happens.

Example

$$(t^4 - 2t^2 + 1)y'' + (t-1)y' + 3y = 0$$

Solution

$$p(t) = \frac{t-1}{(t^2-1)^2} = \frac{1}{(t-1)(t+1)^2} \text{ is unbounded near } t = \pm 1$$

$$q(t) = \frac{3}{(t-1)^2(t+1)^2} \text{ is unbounded near } t = \pm 1$$

$$\text{Singular points are} \pm 1$$

$$\text{For } t_0 = 1:$$

$$(t-1)p(t) = \frac{1}{(t+1)^2}, (t-1)^2q(t) = \frac{3}{(t+1)^2}, \text{both are analytic near 1.}$$

$$\therefore 1 \text{ is a regular singular point}$$

$$\text{For } t_0 = -1:$$

$$(t+1)p(t) = \frac{1}{(t-1)(t+1)} \text{ is unbounded near 1}$$

$$\therefore -1 \text{ is an irregular singular point.}$$

Example

Find a Fundamental Set of Solutions for

$$t^2y'' + ty' + \left(t^2 - \frac{1}{4}\right)y = 0, t > 0$$

Solution

$$p(t) = \frac{1}{t}, q(t) = \frac{t^2 - \frac{1}{4}}{t^2} \Rightarrow \text{regular singularity at } t_0$$

$$y = t^r \sum_{n=0}^{\infty} a_n t^n, a_0 \neq 0$$

$$y = \sum_{n=0}^{\infty} a_n t^{r+n}$$

$$y' = \sum_{n=0}^{\infty} (r+n) t^{r+n-1} \Rightarrow ty' = \sum_{n=0}^{\infty} (r+n) a_n t^{r+n}$$

$$y'' = \sum_{n=0}^{\infty} (r+n)(r+n-1) t^{r+n-2} \Rightarrow t^2 y'' = \sum_{n=0}^{\infty} (r+n)(r+n-1) t^{r+n}$$

$$t^2 y = \sum_{n=0}^{\infty} a_n t^{r+n+2} = \sum_{n=2}^{\infty} a_{n-2} t^{r+n}$$

$$-\frac{1}{4} y = \sum_{n=0}^{\infty} -\frac{1}{4} a_n t^{r+n}$$

$$\begin{aligned}
&\Rightarrow t^2 + ty' + t^2 y - \frac{1}{4} y \\
&= \left(\underbrace{r(r-1)a_0}_{\text{from } ty''} + \underbrace{ra_0}_{\text{from } ty'} - \underbrace{\frac{1}{4} a_0}_{\text{from } -\frac{1}{4} y} \right) t^r \\
&+ \left((r+1)r a_1 + (r+1) a_1 - \frac{1}{4} a_1 \right) t^{r+1} \\
&+ \sum_{n=2}^{\infty} \left((r+n)(r+n-1) a_n + (r+n) a_n - \frac{1}{4} a_n + a_{n-2} \right) t^{r+n} \\
&= 0
\end{aligned}$$

Then we can make equations from that:

$$\left(r(r-1) + r - \frac{1}{4} \right) a_0 = 0 \Rightarrow \left(r^2 - \frac{1}{4} \right) a_0 = 0 \Rightarrow r = \pm \frac{1}{2} \text{ since } a_0 \neq 0$$

$$\left((r+1)r + (r+1) - \frac{1}{4} \right) a_0 = 0 \Rightarrow \left((r+1)^2 - \frac{1}{4} \right) a_1 = 0$$

$$\left((r+n)^2 - \frac{1}{4} \right) a_n + a_{n-2} = 0 \forall n \geq 2$$

Now we can solve this recurrence:

$$\text{take } r = \frac{1}{2}$$

$$\left(\left(\frac{3}{2}\right)^2 - \frac{1}{4}\right)a_1 = 0 \Rightarrow a_1 = 0$$

$$\left(\left(\frac{1}{2} + n\right)^2 - \frac{1}{4}\right)a_n + a_{n-2} = (n + n^2)a_n = -a_{n-2} \Rightarrow a_n = -\frac{a_{n-2}}{n(n+1)}$$

$$a_1 = 0 \Rightarrow a_3 = a_5 = \cdots = 0$$

$$a_2 = -\frac{a_0}{3 \times 2}, a_4 = \frac{a_0}{5 \times 4 \times 3 \times 2} \Rightarrow a_{2n} = \frac{(-1)^n a_0}{(2n+1)!}$$

$$\Rightarrow y = t^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n a_0}{(2n+1)!} t^{2n} = \frac{a_0 \sin(t)}{\sqrt{t}} \Rightarrow \frac{\sin(t)}{\sqrt{t}} \text{ is a solution}$$

$$\text{Take } r = -\frac{1}{2}$$

$$\left(\left(-\frac{1}{2} + 1\right)^2 - \frac{1}{4}\right)a_1 = 0 \Rightarrow 0 = 0$$

$$\text{how unhelpful}$$

$$\text{Assume } a_0 = 0$$

$$\left(\left(-\frac{1}{2} + n\right)^2 - \frac{1}{4}\right)a_n = -a_{n-2} \Rightarrow a_n = \frac{-a_{n-2}}{n(n-1)}$$

$$a_2 = -\frac{a_0}{1 \times 2}, a_4 = \frac{a_1}{1 \times 2 \times 3 \times 4} \Rightarrow a_{2n} = \frac{(-1)^n a_0}{(2n)!}$$

$$a_3 = -\frac{a_1}{3 \times 2} \Rightarrow a_{2n+1} = \frac{(-1)^n a_1}{(2n+1)!}$$

$$y = t^{-\frac{1}{2}} \left(\sum_{n=0}^{\infty} \frac{(-1)^n a_0}{(2n)!} t^{2n} + \sum_{n=0}^{\infty} \frac{(-1)^n a_1}{(2n+1)!} t^{2n+1} \right)$$

$$= a_0 \frac{\cos(t)}{\sqrt{t}} + a_1 \frac{\sin(t)}{\sqrt{t}}$$

$$\therefore \frac{\cos(t)}{\sqrt{t}} \text{ is a solution}$$

Theorem

Let $tp(t) = p_0 + p_1 t + \cdots$ and $t^2 q(t) = q_0 + q_1 t + \cdots$

Then, the indicial equation is $r(r-1) + p_0 r + q_0 = 0$. Let r_1, r_2 be roots with $r_1 \geq r_2$ if $r_1, r_2 \in \mathbb{R}$.

1. If $r_1, r_2 \in \mathbb{R}$:

$$y_1 = t^{r_1}\sum_{n=0}^{\infty} a_n t^n \qquad a_0 \neq 0$$

$$y_2 = cy_1(t)\ln(t) + t^{r_2}\sum_{n=0}^{\infty} b_n t^n \qquad b_0 \neq 0$$

If $r_1 - r_2 \notin \mathbb{Z}$, then $c = 0$

If $r_1 = r_2$, then $c = 1$.

If $r_1 - r_2 \in \mathbb{Z}$, then $c \in \mathbb{R}$

2. If $r_1 = \overline{r_2} = a \pm bi, b \neq 0$:

$$y_1(t) = t^a \cos(b\ln(t))\sum_{n=0}^{\infty} a_n t^n \qquad a_0 \neq 0$$

$$y_2(t) = t^a \sin(b\ln(t))\sum_{n=0}^{\infty} b_n t^n \qquad b_0 \neq 0$$

Proofish

$$y'' + p(t)y' + q(t)y = 0$$

$$tp(t) = p_0 + p_1 t + \cdots, t^2 q(t) = q_0 + q_0 t + \cdots$$

$$y = \sum_{n=0}^{\infty} a_n t^{r+n}, y' = \sum_{n=0}^{\infty}(r+n)a_n t^{r+n-1}, y'' = \sum_{n=0}^{\infty}(r+n)(r+n-1)a_n t^{r+n-2}$$

$$\Rightarrow t^2y'' + t(tp(t))y' + t^2q(t)y = \sum_{n=0}^{\infty}(r+n)(r+n-1)a_n t^{r+n} + (p_0 + p_1 t + \ldots)\sum_{n=0}^{\infty}(r+n)a_n t^{r+n}$$

$$+(q_0 + q_1 t + \cdots)\sum_{n=0}^{\infty} a_n t^{r+n} = 0$$

Form equations:

$$(r(r-1) + p_0 r + q_0)a_0 = 0 \Rightarrow r(r-1) + p_0 r + q_0 = 0$$

Coefficent of t^{r+n}:

$$(r+n)(r+n-1)a_n + \sum_{k=0}^{n} p_k(r+n-k)a_{n-k}\sum_{k=0}^{n} q_k a_{n-k}$$

$$= ((r+n)(r+n-1) + p_0(r+n) + q_0)a_n + \cdots = 0$$

Example

Write down the form of a FSOS at $t_0 = 0$

1. $t^2y'' + (\sin t + t)y' + y = 0, t > 0$
2. $t^2y''(e^t - 1)y' - (t+1)y = 0, t > 0$
3. $t^2y'' + (3t + t^4)y' + y = 0$

Solution

1.

$$p(t)=\frac{\sin(t)+t}{t^2}\Rightarrow tp(t)=\frac{\sin(t)}{t}+1=2-\frac{t^2}{3!}+\frac{t^4}{5!}-\cdots$$
$$q(t)=\frac{1}{t^2}\Rightarrow t^2q(t)=1$$

The indicial equation:

$$r(r-1)+2r+1=0\Rightarrow r=-\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$$

Form of the solutions:

$$y_1=t^{-\frac{1}{2}}\cos\left(\frac{\sqrt{3}}{2}\ln(t)\right)\sum_{n=0}^{\infty}a_nt^n \qquad a_0\neq 0$$
$$y_2=t^{-\frac{1}{2}}\cos\left(\frac{\sqrt{3}}{2}\ln(t)\right)\sum_{n=0}^{\infty}b_nt^n \qquad b_0\neq 0$$

2.

$$p(t)=\frac{e^t-1}{t^2}\Rightarrow tp(t)=\frac{e^t-1}{t}=1+\frac{t}{2!}+\frac{t^2}{3!}+\cdots$$
$$q(t)=-\frac{t+1}{t^2}\Rightarrow t^2q(t)=-1-t$$

Indicial equation:

$$r(r-1)+r-1=0\Rightarrow r^2-1=0\Rightarrow r=\pm 1$$

Form of the equation:

$$y_1=t\sum_{n=0}^{\infty}a_nt^n \qquad a_0\neq 0$$
$$y_2=cy_1\ln(t)+\frac{1}{t}\sum_{n=0}^{\infty}b_nt^n \qquad b_0\neq 0$$

Since $1-(-1)=2$ is a positive integer, c may be any real value.

3.

$$p(t)=\frac{3t+t^4}{t^2}\Rightarrow tp(t)=3+t^3$$
$$q(t)=\frac{1}{t^2}\Rightarrow t^2q(t)=1$$

Indicial equation:

$$r(r-1)+3r+1=0\Rightarrow (r+1)^2=0\to r=-1,-1$$

Form of the equation

$$y_1 = \frac{1}{t}\sum_{n=0}^{\infty} a_n t^n \qquad a_0 \neq 0$$
$$y_2 = y_1 \ln(t) + \frac{1}{t}\sum_{n=0}^{\infty} b_n t^n \qquad b_0 \neq 0$$

Chapter 10: Laplace Transform

We can turn multiplication into addition:

$$ab \xrightarrow{\ln} \ln(a) + \ln(b)$$

This can be useful in calculus for differentiation:

$$\ln\Big((x-1)^3(x+1)^4(x-2)^7\Big) = 3\ln(x-1) + 4\ln(x+1) + 7\ln(x-1)$$

If we could turn differentiation into multiplication, that would be useful:

$$y'' + 3y' + 7y = 0 \xrightarrow{\text{Laplace Transform}} t^2 y + 3ty + 7y = 1$$

Laplace Transform

The Laplace Tranform, $\mathcal{L}$, assigns to any function $f(t)$ defined for all $t \geq 0$ a new function $F(s)$

$$F(s) = \mathcal{L}\{f(t)\}(s) = \int_0^\infty e^{-st} f(t) \mathrm{d}t$$

as long as the integral converges. $\mathcal{L}$ is linear by properties of integrals.

Example

Find the Laplace of the following:

1. e^{at}
2. $e^{at}\cos(bt)$
3. $e^{at}\sin(bt)$

Solution

1.

$$\begin{aligned}\mathcal{L}\{e^{at}\}(s) &= \int_0^\infty e^{-st}e^{at}\mathrm{d}t = \int_0^\infty e^{(a-s)t}\mathrm{d}t \\ &= \begin{cases} \lim_{T\to\infty} \int_0^\infty \mathrm{d}t = \infty & \text{if } a = s \\ \lim_{T\to\infty} \frac{e^{a-s}-1}{a-s} = \infty & \text{if } a \neq s \end{cases} \\ &= \begin{cases} \infty & \text{if } a \geq s \\ \frac{1}{s-a} & \text{if } s > a \end{cases}\end{aligned}$$

2.
3.

Assume $b \neq 0$

$$e^{at}\cos(bt) + ie^{at}\sin(bt) = e^{(a+bi)t}$$

$$\mathcal{L}\{e^{(a+bi)t}\}(s) = \int_0^\infty e^{(a+bi-s)t}\mathrm{d}t = \begin{cases} \text{Diverges} & \text{if } a \geq s \\ \frac{1}{s-a-bi} & \text{if } s > a \end{cases}$$

$$\Rightarrow \mathcal{L}\{e^{(a+bi)t}\}(s) = \frac{s-a+bi}{(s-a)^2+b^2}$$

$$\overset{\text{Re}}{\Rightarrow} \mathcal{L}\{e^{at}\cos(bt)\}(s) = \frac{s-a}{(s-a)^2+b^2}$$

$$\overset{\text{Im}}{\Rightarrow} \mathcal{L}\{e^{at}\sin(bt)\}(s) = \frac{b}{(s-a)^2+b^2}$$

Table of Laplaces

$j(t) = \mathcal{L}^{-1}[J(s)]$	$J(s) = \mathcal{L}[j(t)]$
e^{at}, $a \in \mathbb{R}$	$\frac{1}{s-a}$

What can go wrong with the Laplace

1. f could have too many points of discontinuity
2. f is too large compared to e^{st} For example, $f(t) = e^{t^2} \gg e^{st} \forall s \Rightarrow \int_0^\infty e^{t^2-st}\mathrm{d}t = \infty$

Piecewise continuous

We say $f : [0, \infty) \to \mathbb{R}$ is piecewise continuous if for every $r > 0$ there are only finitely many points of discontinuity for $f(t) \in [0, r]$

Exponential Order

We say $f : [0, \infty) \to \mathbb{R}$ is of exponential order if $\forall t \geq 0, \exists c, M \in \mathbb{R}$ s.t. $|f(t)| \leq Me^{ct}$. We say f is of exponential order not exceeding c.

Examples

1. $|\sin(t)| \leq 1 \therefore \sin(t)$ is of exponential runtime not exceeding 0.
2. t is not of exponential runtime not exceeding 0 because $|t| \leq Me^{0t}$ does not hold for any M. However, $|t| \leq Me^{ct}$ holds for every $c > 0$. M will depend on c.

$$\text{For } \varepsilon = 1 \text{ there is some } s > 0 \text{ such that if } t > s \text{ then } \left|\frac{t}{e^{ct}} - 0\right| < 1$$

$$\Rightarrow \frac{t}{e^{ct}} < 1 \text{ for all } t > s$$

By the EVT, $\frac{t}{e^{ct}}$ attains a max m over $[0, \delta]$. Thus, $\left|\frac{t}{e^t}\right| \leq m$ for all $t \in [0, \delta]$.

$$\left|\frac{t}{e^{ct}}\right| \leq \max(1, m) \text{ for all } t \geq 0$$

$$|t| \leq \underbrace{\max(1, m)e^{ct}}_{m}$$

Theorem on the existence of the Laplace

Suppose $f : [0, \infty] \to \mathbb{R}$ is piecewise continuous and of exponential order. Then its Laplace transformation $\mathcal{L}\{f(t)\}(s)$ exists for sufficiently large s. Specifically, if f is piecewise continuous, and $|f(t)| \leq Me^{ct}$ for constants c, M, and for all $t \geq 0$, then $\mathcal{L}\{f(t)\}(s)$ exists for all $s > c$.

Equality of Laplaces

Suppose $f(t)$ and $g(t)$ are two functions continuous over $[0, \infty)$, both of which are of exponential order. Assume there is a real number A for which $\mathcal{L}\{f(t)\}(s) = \mathcal{L}\{g(t)\}(s)$ for all $s > A$. Then $f(t) = g(t)$ for all $t \in [0, \infty)$.

Essentially, $\mathcal{L}\{f(t)\}(s) = \mathcal{L}\{g(t)\}(s) \iff f(t) = g(t)$ for $s > A, t \geq 0$

Examples

Example

Solve $y' - 2y = e^{5t}, y(0) = 3$:

Solution

We will evaluate the Laplace of y and then take the inverse Laplace.

Suppose $\mathcal{L}\{y(t)\}(s) = Y(s)$.

$$\mathcal{L}\{y'(t)\}(s) - 2Y(s) = \mathcal{L}\{e^{5t}\}$$

$$\mathcal{L}\{y'(t)\}(s) - 2Y(s) = \frac{1}{s-5}$$

$$\mathcal{L}\{y'(t)\}(s) = \int_0^\infty e^{-st}y'(t)\mathrm{d}t = e^{-st}y(t)\Big|_{t=0}^{t=\infty} - \int_0^\infty y(t)(-se^{-st})\mathrm{d}t$$

$$= 0 - \underbrace{y(0)}_{y(0)=3} + sY(s) = sY(s) - 3$$

$$sY(s) - 3 - 2Y(s) = \frac{1}{s-5}$$

$$\Rightarrow Y(s) = \frac{3}{s-2} + \frac{1}{(s-2)(s-5)}$$

$$= \frac{3}{s-2} + \frac{-\frac{1}{3}}{s-2} + \frac{\frac{1}{3}}{s-5}$$

$$= y(t) = \mathcal{L}^{-1}\left\{\frac{8}{3(s-2)} + \frac{1}{3(s-5)}\right\}(t) = \frac{8}{3}e^{2t} + \frac{1}{3}e^{5t}$$

Theorem on the order of a solution

Suppose $f : [0, \infty) \to \mathbb{R}$ is of exponential order. Let $L[y] = f(t)$ be a linear equation with constant coefficients. Then, every solution to this equation is of exponential order.

Theorem on the Laplace of a derived function

Suppose $f : [0, \infty) \to \mathbb{R}$ is n times differentiable, $f^{(n)}(t)$ is piecewise continuous, and of exponential order not exceeding c. Let $F(s) = \mathcal{L}\{f(t)(s)\}$. Then,

$$\mathcal{L}\left\{f^{(n)}(t)\right\}(s) = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - f^{(n-1)}(0)$$

Proof

We know

$$\left|f^{(n)}(t)\right| \le Me^{ct} \Rightarrow -Me^{ct} \le f^{(n)} \le Me^{ct}$$
$$\overset{\text{integration}}{\Rightarrow} \quad -\frac{Me^{ct}}{c} + \frac{Me^0}{c} \le f^{(n-1)}(t) - f^{(n-1)}(0) \le \frac{Me^{ct}}{c} - \frac{Me^0}{c}$$

Therefore, $f^{(n-1)}$ is of exponential order.

$$\mathcal{L}\left\{f^{(n)}(t)\right\}(s) = \int_0^\infty e^{-st} f^{(n)}(t)\mathrm{d}t = e^{-st} f^{(n-1)} - \int_0^\infty f^{(n-1)}(t)\left(-se^{-st}\right)\mathrm{d}t$$
$$= 0 - e^0 f^{(n-1)}(0) + s\mathcal{L}\left\{f^{(n-1)}(t)\right\}(s)$$

Finish off the proof by applying the inductive hypothesis.

How to find a particular solution

Suppose we want to find a particular solution for $y^{(n)} + a_n y^{(n-1)} + \dots + a_1 y = f(t)$, where $a_1, \dots, a_n$ are constants.

Let $\mathcal{L}\{f(t)\}(s) = F(s)$ and $\mathcal{L}\{y(t)\} = Y(s)$

$$\begin{aligned} & y^{(n)} + a_n y^{(n-1)} + \dots + a_1 y = f(t) \\ \Rightarrow\ & s^n Y(s) = s^n Y(s) - s^{n-1} y(0) - s^{n-2} y'(0) - \dots - y^{(n-1)}(0) \\ & + a_n\left(s^{n-1} Y(s) - s^{n-2} y(0) - \dots - y^{(n-2)}(0)\right) \\ & + \dots + a_1 Y(s) = F(s) \end{aligned}$$

Set $y(0) = \dots = y^{(n-1)}(0) = 0$

$$s^n Y(s) + a_n s^{n-1} Y(s) + \dots + a_1 Y(s) = F(s)$$
$$Y(s) = \frac{F(s)}{s^n + a_n s^{n-1} + \dots + a_1} = \frac{F(s)}{\underbrace{p(s)}_{\text{characteristic polynomial}}}$$
$$Y_p = \mathcal{L}^{-1}\left\{\frac{F(s)}{p(s)}\right\}(t)$$

Theorem on derivatives

Suppose $f : [0, \infty) \to \mathbb{R}$ is piecewise continuous and of exponential order not exceeding c. Then its Laplace transform $F(s)$ is infinitely differentiable and for every positive integer n and every real number a, we have

1. $\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s)$, for all $s > c$
2. $\mathcal{L}\left\{e^{at} f(t)\right\} = F(s - a)$, for all $s > a + c$

Proof

1.

$$F(s)=\mathcal{L}\{f(t)\}(s)=\int_0^\infty e^{-st}f(t)\mathrm{d}t$$

$$F'(s)=\int_0^\infty -te^{-st}f(t)\mathrm{d}t=-\int_0^\infty e^{-st}tf(t)\mathrm{d}t=-\mathcal{L}\{tf(t)\}(s)$$

$$\Rightarrow \mathcal{L}\{t^n f(t)\}(s)=(-1)F^{(n)}(s)$$

2.

$$\mathcal{L}\{e^{at}f(t)\}(s)=\int_0^\infty e^{-st}e^{at}f(t)\mathrm{d}t=\int_0^\infty e^{-(s-a)t}f(t)\mathrm{d}t=\mathcal{L}\{f(t)\}(s-a)=F(s-a)$$

Examples

Example
Find the inverse Laplace of $\frac{1}{(s+1)^2}$

Solution

$$\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}(t)=e^{-t}$$

$$\mathcal{L}\{te^{-t}\}(s)=-\frac{\mathrm{d}}{\mathrm{d}s}(\mathcal{L}\{e^{-t}\}(s))=-\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{s+1}\right)=\frac{1}{(s+1)^2}\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\}(t)=te^{-t}$$

Example
Using the Laplace transform, solve the IVP:

$$y'''+2y''+y'=0, y(0)=1, y'(0)=y''(0)=0$$

Solution
Let $Y(s)=\mathcal{L}(y(t))(s)$.

Take the Laplace transform of both sides

$$\underbrace{s^3Y(s)-s^2y(0)-sy'(0)-y''(0)}_{\mathcal{L}\{y'''\}}+\underbrace{2s^2Y(s)-2sy(0)-2y'(0)}_{2\mathcal{L}\{y''\}}+\underbrace{sY(s)-y(0)}_{\mathcal{L}\{y'\}}=\underbrace{0}_{\mathcal{L}\{0\}}$$

$$(s^3+2s^2+s)Y(s)-s^2-2s-1=0\Rightarrow Y(s)=\frac{1}{s}\Rightarrow y(t)=\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}(t)=e^{0t}=1$$

$$\boxed{y(t)=1}$$

Example

$$\mathcal{L}^{-1}\left\{\frac{5}{s^4+13s^2+36}\right\}$$

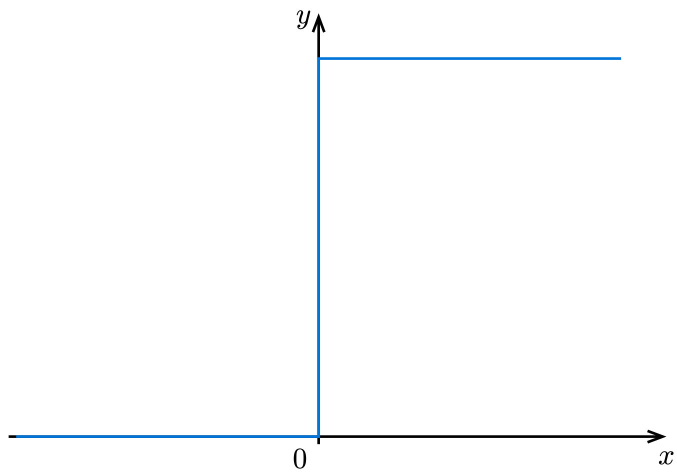
Solution

$$\frac{5}{s^4+13s^2+36} = \frac{1}{s^2+4} - \frac{1}{s^2+9}$$
$$\mathcal{L}^{-1}\left\{\frac{5}{s^4+13s^2+36}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\}$$
$$= \boxed{\frac{1}{2}\sin(2t) + \frac{1}{3}\sin(3t)}$$

Definition of Heaviside

$$H(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

And here is a plot of $H(t)$:



Example

$$f(t) = \begin{cases} \sin(t) & t \geq 0 \\ 0 & t \leq 0 \end{cases}$$

This can be written as:

$$f(t) = H(t)\sin(t)$$

You can write down ones that aren't at zeros by shifting the Heaviside function;

$$f(t) = \begin{cases} \sin(t) & t \geq 1 \\ 0 & t \leq 1 \end{cases} = H(t-1)\sin(t)$$

If it only exists in a particular region, you can deal with that by subtracting them:

$$\begin{cases} 0 & \text{if } t < 1 \\ 1 & \text{if } 1 \leq t < 2 = H(t-1) - H(t-2) \\ 0 & \text{if } 2 \leq t \end{cases}$$

Example

$$\mathcal{L}\{H(t-c)f(t-c)\}(s) \text{ for } c \geq 0$$

Solution

$$\int_0^\infty e^{-st}H(t-c)f(t-c)\mathrm{dt} = \int_c^\infty e^{-st}H(t-c)f(t-c)\mathrm{dt}$$

$$\overset{u=t-c}{=} \int_0^\infty e^{-s(u+c)}f(u)\mathrm{du} = e^{-sc}\int_0^\infty e^{-su}f(u)\mathrm{du} = e^{-sc}\mathcal{L}\{f(t)\}(s)$$

Example

Find the Laplace of $f(t)$

$$f(t) = \begin{cases} t^2 & 0 \leq t < 2 \\ 1-t & 2 \leq t < 3 \\ 1 & 3 \leq t \end{cases}$$

Solution

Rewrite with Heaviside functions:

$$f(t) = t^2(H(t) - H(t-2)) + (1-t)(H(t-2) - H(t-3)) + 1(H(t-3))$$

Recall:

$$\mathcal{L}\{H(t-c)j(t-c)\}(s) = e^{-cs}\mathcal{L}\{j(t)\}(s)$$

Rewrite:

$$f(t) = H(t)\underbrace{t^2}_{j_1} + H(t-2)\underbrace{(-t^2+1-t)}_{j_2} + H(t-3)\underbrace{(-1+t+1)}_{j_3}$$

$$j_1(t) = t^2$$
$$j_2(t-2) = -t^2+1-t \Rightarrow j_2(t) = -t^2-5t-5$$
$$j_3(t-3) = t \Rightarrow j_3(t) = t+3$$

$$\mathcal{L}\{H(t)t^2\}(s) = e^{-0s}\mathcal{L}\{t^2\}(s) = \frac{2!}{s^3} = \frac{2}{s^3}$$
$$\mathcal{L}\{H(t-2)j_2(t-2)\}(s) = e^{-2s}\mathcal{L}\{-t^2-5t-5\}(s) = e^{-2s}\left(-\frac{2}{s^3} - \frac{5}{s^2} - \frac{5}{s}\right)$$
$$\mathcal{L}\{H(t-3)j_3(t-3)\}(s) = e^{-3s}\mathcal{L}\{t+3\}(s) = e^{-3s}\left(\frac{1}{s^2} + \frac{3}{s}\right)$$

Therefore, after summing up:

$$\mathcal{L}\{f(t)\}(s) = \frac{2}{s^3} - e^{-2s}\left(\frac{2}{s^3} + \frac{5}{s^2} + \frac{5}{s}\right) + e^{-3s}\left(\frac{1}{s^2} + \frac{3}{s}\right)$$

Convolution

$$(f \star g)(t) = \int_0^t f(x)g(t-x)\mathrm{d}x$$

$$f \star g = g \star f$$

Product of Laplaces

$$\mathcal{L}\{f\}\mathcal{L}\{g\} = \mathcal{L}\{f \star g\}$$

Proofish

$$\mathcal{L}\{f \star g\} = \int_0^\infty e^{-st}(f \star g)(t)\mathrm{d}t$$

$$= \int_0^\infty e^{-st} \int_0^t f(x)g(t-x)\mathrm{d}x\mathrm{d}t$$

$$= \int_0^\infty \int_0^t e^{-st} f(x)g(t-x)\mathrm{d}x\mathrm{d}t$$

$$\overset{\text{questionable}}{=} \int_0^\infty \int_x^\infty e^{-st} f(x)g(t-x)\mathrm{d}t\mathrm{d}x$$

$$= \int_0^\infty f(x) \int_x^\infty e^{-st} g(t-x)\mathrm{d}t\mathrm{d}x$$

$$\overset{u=t-x}{=} \int_0^\infty f(x) \int_x^\infty e^{-s(u+x)} g(u)\mathrm{d}t\mathrm{d}x$$

$$= \int_0^\infty f(x)e^{-sx} \int_x^\infty e^{-su} g(u)\mathrm{d}t\mathrm{d}x$$

$$= \int_0^\infty f(x)e^{-sx} \mathcal{L}\{g\}(s)\mathrm{d}t\mathrm{d}x$$

$$= \mathcal{L}\{g\}(s) \int_0^\infty f(x)e^{-sx}\mathrm{d}t\mathrm{d}x$$

$$= \mathcal{L}\{g\}(s)\mathcal{L}\{f\}(s)$$

Example

$$\mathcal{L}^{-1}\left\{\frac{1}{s^4 + 2s^2 + 1}\right\}$$

Solution

$$\frac{1}{s^4 + 2s^2 + 1} = \frac{1}{\left(s^2+1\right)^2} = \frac{1}{s^2+1} \cdot \frac{1}{s^2+1}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}(s) = \sin(t)$$

$$(\sin \star \sin)(t) = \int_0^t \sin(x)\sin(t-x)\mathrm{d}x$$

$$= \int_0^t \frac{1}{2}(\cos(t-2x) - \cos(t))\mathrm{d}x = \cdots \square$$

Another solution

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin(t)$$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos(t)$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{s^2+1}\right)\right\} = -t\sin(t)$$

$$\mathcal{L}^{-1}\left\{\frac{-2s}{\left(s^2+1\right)^2}\right\} = -t\sin(t)$$

$$\text{similarly } \mathcal{L}^{-1}\left\{\frac{s^2+1-2s^2}{\left(s^2+1\right)^2}\right\} = -t\cos(t)$$

$$\mathcal{L}^{-1}\left\{\frac{2-s^2-1}{\left(s^2+1\right)^2}\right\} = -t\cos(t)$$

$$\mathcal{L}^{-1}\left\{\frac{2}{\left(s^2+1\right)^2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = -t\cos(t)$$

$$\mathcal{L}^{-1}\left\{\frac{2}{\left(s^2+1\right)^2}\right\} - \sin(t) = -t\sin(t) \Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{\left(s^2+1\right)^2}\right\} = \frac{1}{2}(\sin(t) - t\cos(t))$$

Chapter 11: Systems of Differential Equations

First-Order System

$x_1(t), ..., x_n(t)$ are unknown.

$$\begin{cases} \frac{\mathrm{d}x_1}{\mathrm{d}t} = f_1(t, x_1, ..., x_n) \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} = f_2(t, x_2, ..., x_n) \\ \vdots \\ \frac{\mathrm{d}x_n}{\mathrm{d}t} = f_n(t, x_2, ..., x_n) \end{cases}$$

We use $\vec{x} = (x_1, ..., x_n)$; $\vec{f} = (f_1, ..., f_n)$, and write $\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = \vec{f}(t, \vec{x})$ as the compact form. $\vec{f}$ is called forcing.

We can write down any differential equation or system as a first-order system by using something like $x_1 = x, x_2 = x'$, etc.

Example

Convert the system into a first-order system:

$$\begin{cases} x'' = x^2 + x' + txy' \\ y'' = y'y + yt^3 \end{cases}$$

Solution

$$x_1 = x_1, x_2 = x', x_3 = y, x_4 = y'$$

$$\begin{cases} \frac{\mathrm{d}x_1}{\mathrm{d}t} = x_2 \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} = x_1^2 + x_2 + tx_1x_4 \\ \frac{\mathrm{d}x_3}{\mathrm{d}t} = x_4 \\ \frac{\mathrm{d}x_4}{\mathrm{d}t} = x_4x_3 + t^3x_3 \end{cases}$$

Linear Systems

$$\begin{cases} \frac{\mathrm{d}x_1}{\mathrm{d}t} = a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + f_1(t) \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} = a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n + f_2(t) \\ \vdots \\ \frac{\mathrm{d}x_n}{\mathrm{d}t} = a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + f_n(t) \end{cases}$$

This can also be written in matrix form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ & \vdots & & \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix}$$

Every first-order linear system can be written as $\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = A(t)\vec{x} + \vec{f}(t)$, where $A(t)$ is an $n \times n$ matrix with entries as functions of t, $\vec{f}(t)$ is a $n \times 1$ column whose entries are functions of t, and $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

Example

Find the forcing and coefficient matrix of:

$$\begin{cases} x_1' = 2x_1 - tx_2 + \sin(t) \\ x_2' = t^2x_1 + \cos(t)x_2 \end{cases}$$

Solution

$$A(t) = \begin{pmatrix} 2 & -t \\ t^2 & \cos(t) \end{pmatrix}$$

$$\vec{f}(t) = \begin{pmatrix} \sin(t) \\ 0 \end{pmatrix}$$

Example
Write as a first-order system and find the coefficient matrix and the forcing.

$$y''' - y'' + ty' + \tan(t)y = e^{t^2}$$

Solution

$$x_1 = y, x_2 = y', x_3 = y''$$

$$\begin{cases} \frac{\mathrm{d}x_1}{\mathrm{d}t} = x_2 \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} = x_3 \\ \frac{\mathrm{d}x_3}{\mathrm{d}t} = x_3 - tx_2 - \tan(t)x_1 + e^{t^2} \end{cases}$$

The coeffient matrix:

$$A(t) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\tan(t) & -t & 1 \end{pmatrix}$$

The forcing:

$$\vec{f}(t) = \begin{pmatrix} 0 \\ 0 \\ e^{t^2} \end{pmatrix}$$

Example
Convert the following nth order linear equation into a first-order linear system. Find its coefficient matrix and forcing.

$$y^{(n)} + a_n(t)y^{(n-1)} + \cdots + a_1(t)y = f(t)$$

Solution
Set:

$$x_1 = y, x_2 = y', ..., x_n = y^{(n-1)}$$

Then:

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ \vdots \\ x_{n-1}' = x_n \\ x_n' = -a_1(t)x_1 - a_2(t)x_2 - \cdots - a_n(t)x_n + f(t) \end{cases}$$

The coefficient matrix is therefore:

$$A(t) = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & \cdots & & & 0 & 1 \\ -a_1(t) & -a_2(t) & -a_3(t) & \cdots & -a_{n-1}(t) & -a_{n(t)} \end{pmatrix}$$

And the forcing is:

$$\vec{f}(t) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ f(t) \end{pmatrix}$$

To solve a nth order linear differential equation

Similar to nth order linear differential equation, to solve $\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = A(t)\vec{x} + \vec{f}(t)$

1. Find the general solution $\vec{x}_H$ to $\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = A(t)\vec{x}$
2. Find a particular solution $\vec{x}_p$ to $\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = A(t)\vec{x} + \vec{f}(t)$
3. The general solution to $\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = A(t)\vec{x} + \vec{f}(t)$ is $\vec{x}_H + \vec{x}_p$

We always assume this process is done over an interval unless specified otherwise.

Existence and Uniqueness theorem for First-Order Differential Equations

$$\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = A(t)\vec{x} + \vec{f}(t); A(t) \text{ is } n \times n$$

All entries are continuous over (a, b) and $t_0 \in (a, b)$.

Then, this has a unique solution defined over (a, b):

$$\begin{cases} \frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = A(t)\vec{x} + \vec{f}(t) \\ \vec{x}(t_0) = \vec{x}_0 \end{cases}$$

Example

Find the largest interval that the IVP has a unique solution over:

$$\begin{cases} t^2 x' = 2x - (\cos(t))y + \tan(t) \\ (\sin t)y' = tx + y + \cos(t) \\ x(1) = 0 \\ y(1) = 0 \end{cases}$$

Solution

$$A(t) = \begin{pmatrix} \frac{2}{t^2} & -\frac{\cos(t)}{t^2} \\ \frac{t}{\sin(t)} & \frac{1}{\sin(t)} \end{pmatrix}$$

$$\vec{f}(t) = \begin{pmatrix} \frac{\tan(t)}{t^2} \\ \cot(t) \end{pmatrix}$$

We need all entries to be continuous, so

$$t \neq 0, t \neq k\pi, t \neq \frac{\pi}{2} + j\pi \qquad (k, j \in \mathbb{Z})$$

Because $t_0 = 1$ is in the interval, and $\frac{\pi}{2} > 1$, we are in the first range between 0 and $\frac{\pi}{2}$. Therefore, $\left(0, \frac{\pi}{2}\right)$ is the largest interval.

Dimension of First-order Homogenous Linear System

Consider the first-order n-dimensional system $\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = A(t)\vec{x}$. If all entries of $A(t)$ are continuous over (a, b), then the solution set, defined over (a, b), is an n-dimensional vector space.

Proof

Proof of subspace

$$\vec{x} = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} \subset C(a, b) \times \cdots \times C(a, b)$$

The solution set is a subset of $(C(a, b))^n$. Then apply the subspace criterion.

Proof of subspace, alternative

$$L(\vec{x}) = \frac{\mathrm{d}\vec{x}}{\mathrm{d}t} - A(t)\vec{x} \text{ is linear}$$

Thus, its kernel is a subspace.

Proof of Basis

Fix $t_0 \in (a, b)$

$$\exists \text{ a solution } \overrightarrow{N_j} \text{ to } \begin{cases} \frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = A(t)\vec{x} \\ \vec{x}(t_0) = \vec{e_j} \end{cases} \text{ for } j = 1, ..., n$$

We will show $\overrightarrow{N_1}, ..., \overrightarrow{N_n}$ form a basis for the solution set.

Linear Independence

Suppose $\mathbb{c}_1, ..., \mathbb{c}_n \in \mathbb{R}$ satisfy $\mathbb{c}_1\overrightarrow{N_1}(t) + \cdots + \mathbb{c}_n\overrightarrow{N_n}(t) = \vec{0}$ for all $t \in (a, b)$. Substitute $t = t_0$ to obtain $\mathbb{c}_1\vec{e_1} + \cdots + \mathbb{c}_n\vec{e_n} = \vec{0}$. By linear independence, $\mathbb{c}_1 = \cdots = \mathbb{c}_n = 0$, which shows that $\overrightarrow{N_1}, ..., \overrightarrow{N_n}$ is linear independent.

Spanning

Suppose $\vec{y}(t)$ is a solution defined over (a, b) to $\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = A(t)\vec{x}$.

$\vec{y}(t)$ satisfies:

$$\begin{cases} \frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = A(t)\vec{x} \\ \vec{x}(t_0) = \vec{y}(t_0) \end{cases}$$

Assume $\vec{y}(t_0) = (y_1, ..., y_n)$. Since the solution set is a vector space, $y_1\overrightarrow{N_1}(t) + \cdots + y_n\overrightarrow{N_n}(t)$ is a solution to $\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = A(t)\vec{x}$.

$$y(t_0) = (y_1, ..., y_n) = y_1\vec{e_1} + \cdots + y_n\vec{e_n} = y_1\overrightarrow{N_1}(t_0) + \cdots + y_n\overrightarrow{N_n}(t_0)$$

Therefore, $y_1\overrightarrow{N_1}(t) + \cdots + y_n\overrightarrow{N_n}(t)$ satisifies $\vec{x}(t_0) = \vec{y}(t_0)$. Because $\vec{y}(t)$ could be any solution, and this can represent any of those solutions, this set spans the solution set.

Wronskian

The Wronskian of n functions is defined and denoted by:

$$W\left[\vec{\phi}_1(t), ..., \vec{\phi}_n(t)\right] = \det\left(\vec{\phi}_1(t) \ \cdots \ \vec{\phi}_n(t)\right)$$

Wronskian shows when a solution works

If $W\left[\vec{\phi}_1(t), ..., \vec{\phi}_n(t)\right]\Big|_{t=t_0} = 0$, then $\vec{\phi}_1(t), ..., \vec{\phi}_n(t)$ form a basis for the solution set of $\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = A(t)\vec{x}$.

Proof

$$\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = A(t)\vec{t} \qquad \text{with } t \in (a,b), A \text{ is } n \times n$$

How do we know solutions $\vec{\phi}_1(t), ..., \vec{\phi}_n(t)$ form a basis for the solution set of $\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = A(t)\vec{x}$? Fix $t_0 \in (a,b)$. We know every solution to $\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = A(t)\vec{x}$ is a solution to some IVP:

$$\begin{cases} \frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = A(t)\vec{x} \\ \vec{x}(t_0) = \vec{c_0} \end{cases}$$

In other words, we need to ensure that every solution to the IVP is a linear combination of $\vec{\phi}_1(t), ..., \vec{\phi}_n(t)$.

By linearity $\forall \mathbb{c}_1, ..., \mathbb{c}_n \in \mathbb{R}, \mathbb{c}_1\vec{\phi}_1(t) + \cdots + \mathbb{c}_n\vec{\phi}_n(t)$ satisfies $\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = A(t)\vec{x}$.

We need to make sure for every $\vec{c_0} \in \mathbb{R}^n, \exists \mathbb{c}_1, ..., \mathbb{c}_n \in \mathbb{R}$ such that $\mathbb{c}_1\vec{\phi}_1(t_0) + \cdots + \mathbb{c}_n\vec{\phi}_n(t_0) = \vec{c_0}$.

This can also be written with matrices:

$$\underbrace{\left(\vec{\phi}_1(t_0) \ \cdots \ \vec{\phi}_n(t_0)\right)}_{n \times n}\begin{pmatrix} \mathbb{c}_1 \\ \mathbb{c}_2 \\ \vdots \\ \mathbb{c}_n \end{pmatrix} = \vec{c_0}$$

This matrix equation has a solution $\mathbb{c}_1, ..., \mathbb{c}_n$ for every $\vec{c_0} \in \mathbb{R}^n$ iff

$$\det\left(\vec{\phi}_1(t_0) \ \cdots \ \vec{\phi}_n(t_0)\right) \neq 0$$

Relation to other Wronskian

$$y^{(n)} + a_n(t)y^{(n-1)} + \cdots + a_1(t)y = 0$$

$$Y_1, \ldots, Y_n \text{ are solutions} \Rightarrow W[Y_1, \ldots, Y_n](t) = \det\begin{pmatrix} Y_1(t) & \cdots & Y_n(t) \\ Y_1'(t) & \cdots & Y_n'(t) \\ & \vdots & \\ Y_1^{(n-1)}(t) & \cdots & Y_n^{(n-1)}(t) \end{pmatrix}$$

But also, this equation becomes

$$\begin{cases} \frac{\mathrm{d}x_1}{\mathrm{d}t} = x_2 \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} = x_3 \\ \vdots \\ \frac{\mathrm{d}x_n}{\mathrm{d}t} = -a_n(t)x_{n-1} - \cdots - a_1(t)x_1 \end{cases}$$

$$\overrightarrow{\phi_k}(t) = \begin{pmatrix} Y_k \\ Y_k' \\ \vdots \\ Y_k^{(n-1)} \end{pmatrix}$$

Then:

$$W\left[\vec{\phi}_1, \ldots, \vec{\phi}_n\right] = \det\begin{pmatrix} Y_1(t) & \cdots & Y_n(t) \\ Y_1'(t) & \cdots & Y_n'(t) \\ & \vdots & \\ Y_1^{(n-1)}(t) & \cdots & Y_n^{(n-1)}(t) \end{pmatrix} = W[Y_1, \ldots, Y_n]$$

Wronskian is never zero if it isn't zero at a point

Suppose all entries of the coefficient matrix of a first-order n-dimensional homogenous linear system are continuous over (a, b). Suppose the Wronskian of n solutions to this system is zero at one point $t_0 \in (a, b)$, then the Wronskian must be zero everywhere on (a, b)

$$W[\ldots](t_0) \neq 0 \Rightarrow \forall t \in (a, b), W[\ldots](t) \neq 0$$

Proof

Since $\det\left(\vec{\phi}_1(t_0) \ \ldots \ \vec{\phi}_n(t_0)\right) \neq 0$, by the proof above, $\vec{\phi}_1(t), \ldots, \vec{\phi}_n(t)$ form a basis.

We will show $\det\left(\vec{\phi}_1(t) \ \ldots \ \vec{\phi}_n(t)\right) \neq 0$ for all $t \in (a, b)$.

On the contrary, assume $\det\left(\vec{\phi}_1(t_1) \ \ldots \ \vec{\phi}_n(t_1)\right) = 0$ for some $t_1 \in (a, b)$.

By that assumption $\vec{\phi}_1(t_1), \ldots, \vec{\phi}_n(t_1)$ are linearly dependent. This then means $\mathbb{c}_1\vec{\phi}_1(t_1) + \cdots + \mathbb{c}_n\vec{\phi}_n(t_1) = \vec{0}$ for some $\mathbb{c}_1, \ldots, \mathbb{c}_n \in \mathbb{R}$ not all zero.

Then, $\vec{\varphi}(t) = \mathbb{c}_1\vec{\phi}_1(t) + \cdots + \mathbb{c}_n\vec{\phi}_n(t)$ satisfies the IVP:

$$\begin{cases} \frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = A(t)\vec{x} \\ \vec{x}(t_1) = \vec{\varphi}(t_1) = \vec{0} \end{cases}$$

Since $\vec{0}$ is a solution to the IVP, by the existence and uniqueness theorem, $\vec{\varphi}(t) = \mathbb{c}_1\vec{\phi}_1(t) + \cdots + \mathbb{c}_n\vec{\phi}_n(t) = \vec{0}$ for all $t \in (a, b)$.

Therefore:

$$\left(\vec{\phi}_1(t) \;\cdots\; \vec{\phi}_n(t)\right)\begin{pmatrix} \mathbb{c}_1 \\ \vdots \\ \mathbb{c}_n \end{pmatrix} = \vec{0} \Rightarrow W\left[\vec{\phi}_1(t), \cdots, \vec{\phi}_n(t)\right] = 0$$

for all $t \in (a, b)$, a contradiction.

Abel's Theorem

If W is the Wronskian of n solutions to $\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = A(t)\vec{x}$, then $\frac{\mathrm{d}W}{\mathrm{d}t} = \mathrm{tr}(A(t))W$.

Fundamental Matrix

$$\left(\vec{\phi}_1(t) \;\cdots\; \vec{\phi}_n(t)\right)$$

Solutions from Fundamental Matrix

Assuming $\vec{\phi}_1(t), \cdots, \vec{\phi}_n(t)$ form a FSOS, every solution is of the form:

$$\mathbb{c}_1\vec{\phi}_1(t) + \cdots + \mathbb{c}_n\vec{\phi}_n(t) = \underbrace{\left(\vec{\phi}_1(t) \;\cdots\; \vec{\phi}_n(t)\right)}_{\text{Fundamental Matrix}} \underbrace{\begin{pmatrix} \mathbb{c}_1 \\ \vdots \\ \mathbb{c}_n \end{pmatrix}}_{\text{some vector in } \mathbb{R}^n}$$

Example

Suppose

$$\vec{x_1}(t) = \begin{pmatrix} 1 + t^2 \\ t \end{pmatrix}, \vec{x_2}(t) = \begin{pmatrix} t \\ 1 \end{pmatrix}$$

are two solutions to a first-order 2-dimensional linear system $\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = A(t)\vec{x}$.

1. Find the coefficient matrix $A(t)$.
2. Find the general solution.
3. Find a fundamental matrix.

Solution

1.

$$\begin{pmatrix} 2t \\ 1 \end{pmatrix} = A(t)\begin{pmatrix} 1 + t^2 \\ t \end{pmatrix}$$
$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = A(t)\begin{pmatrix} t \\ 1 \end{pmatrix}$$

Therefore, by the definition of matrix multiplication:

$$\begin{pmatrix} 2t & 1 \\ 1 & 0 \end{pmatrix} = A(t) \begin{pmatrix} 1+t^2 & t \\ t & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1+t^2 & t \\ t & 1 \end{pmatrix}^{-1} = \underbrace{\frac{1}{1+t^2-t^2}}_{1} \begin{pmatrix} 1 & -t \\ -t & 1+t^2 \end{pmatrix}$$

Then,

$$A(t) = \begin{pmatrix} 2t & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -t \\ -t & 1+t^2 \end{pmatrix}$$

2.

Note that all entries of $A(t)$ are continuous over $\mathbb{R}$.

$$W\left[\begin{pmatrix} 1+t^2 \\ 1 \end{pmatrix}, \begin{pmatrix} t \\ 1 \end{pmatrix}\right] = 1 + t^2 - t^2 = 1 \neq 0$$

By a theorem, the general solution is

$$\mathbb{c}_1 \begin{pmatrix} 1+t^2 \\ 1 \end{pmatrix} + \mathbb{c}_n \begin{pmatrix} t \\ 1 \end{pmatrix}$$

3.

A fundamental matrix is

$$\begin{pmatrix} 1+t^2 & t \\ t & 1 \end{pmatrix}$$

Chapter 12: Linear Systems with Constant Coefficients

Homogenous Linear Systems with Constant Coefficients

$$\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = A\vec{x}, A \in M_n(\mathbb{R})$$

$$e^{tA} = I + \frac{tA}{1!} + \frac{t^2A^2}{2!} + \frac{t^3A^3}{3!} + \cdots + \frac{t^nA^n}{n!} + \cdots \qquad \forall t \in \mathbb{R}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{tA}\right) = 0 + A + \frac{2tA^2}{2!} + \frac{3t^2A^3}{3!} + \cdots + \frac{nt^{n-1}A^n}{n!} + \cdots$$
$$= \sum_{n=1}^{\infty} \frac{nt^{n-1}A^n}{n!} = \sum_{n=1}^{\infty} \frac{t^{n-1}A^n}{(n-1)!} = \sum_{n=0}^{\infty} \frac{t^nA^{n+1}}{n!} = A\sum_{n=0}^{\infty} \frac{t^nA^n}{n!} = Ae^{tA}$$

We can form solutions here:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{tA}\right) = Ae^{tA}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(j\text{th column of } e^{tA}\right) = A\left(j\text{th column of } e^{tA}\right)$$

Therefore, the columns of e^{tA} are solutions to $\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = A\vec{x}$.

The Wronskian of these solutions is $\det(e^{tA})$. At $t = 0$, $\det(e^{0A}) = \det(I) = 1 \neq 0$. Therefore, e^{tA} is a fundamental matrix.

Then, this makes $\vec{x}(t) = e^{tA}\vec{\mathbb{c}}$, $\vec{\mathbb{c}} \in \mathbb{R}^n$ the solution.

How do we evaluate e^{tA}?

Set $D = \frac{\mathrm{d}}{\mathrm{d}t}$.

$$D[t] = Ae^{tA}, D^2\left[e^{tA}\right] = A^2 e^{tA}, ..., D^k\left[e^{tA}\right] = A^k e^{tA}$$

For every polynomial $p(z)$ we have $P(D)\left[e^{tA}\right] = p(A)e^{tA}$.

If $p(A) = 0$, then $p(D)\left[e^{tA}\right] = 0$. Therefore, entries of e^{tA} satisfy the differential equation $P(D)[y] = 0$.

In order to find these entries, we need initial values.

If the degree of p is m, we need m initial values: $y(0), y'(0), ..., y^{(m-1)}(0)$.

These will then be:

$$e^{tA}\big|_{t=0} = I; \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{tA}\right)\big|_{t=0} = A, ..., \frac{\mathrm{d}^{m-1}}{\mathrm{d}t^{m-1}}\left(e^{tA}\right)\big|_{t=0} = A^{m-1}$$

Suppose $N_0(t), ..., N_{m-1}(t)$ form the NFSoS for $p(D)[y] = 0$ at $t_0 = 0$.

Then, $e^{tA} = N_0(t)I + N_1(t)A + \cdots + N_{m-1}(t)A^{m-1}$.

Example
Compute e^{tA} where $A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$. Use this to solve $\vec{x} = A\vec{x}, \vec{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Solution
The characteristic polynomial of A is:

$$p(z) = \det\begin{pmatrix} 3 - z & 2 \\ 2 & 3 - z \end{pmatrix} = (3 - z)^2 - 4$$

By the Cayley-Hamilton theorem, $p(A) = 0$, so we can use the process outlined above.

Find the NFSoS for $p(D)[y] = 0$:

$$(3 - z)^2 - 4 = 0 \Rightarrow z \in \{1, 5\} \Rightarrow y = \mathbb{c}_1 e^t + \mathbb{c}_2 e^{5t}$$

$$\begin{cases} y(0) = y_0 \\ y'(0) = y_1 \end{cases} \Rightarrow \begin{cases} \mathbb{c}_1 + \mathbb{c}_2 = y_0 \\ \mathbb{c}_1 + 5\mathbb{c}_2 = y_1 \end{cases} \Rightarrow \begin{cases} \mathbb{c}_1 = \frac{5y_0 - y_1}{4} \\ \mathbb{c}_2 = \frac{y_1 - y_0}{4} \end{cases}$$

$$\Rightarrow y = \left(\frac{5y_0 - y_1}{4}\right)e^t + \left(\frac{y_1 - y_0}{4}\right)e^{5t}$$
$$= y_0\left(\frac{5e^t - e^{5t}}{4}\right) + y_1\left(\frac{e^{5t} - e^t}{4}\right)$$

$$e^{tA} = \left(\frac{5e^t - e^{5t}}{4}\right) I + \left(\frac{e^{5t} - e^t}{4}\right) A$$
$$= \begin{pmatrix} \frac{1}{4}(5e^t - e^{5t}) & 0 \\ 0 & \frac{1}{4}(5e^t - e^{5t}) \end{pmatrix} + \begin{pmatrix} \frac{3}{4}(e^{5t} - e^t) & \frac{1}{2}(e^{5t} - e^t) \\ \frac{1}{2}(e^{5t} - e^t) & \frac{3}{4}(e^{5t} - e^t) \end{pmatrix}$$

The general solution is $\vec{x} = e^{tA}\vec{\mathbb{c}}$. The solution to the given IVP satisfies $\vec{x}(0) = \binom{1}{2} \Rightarrow e^{0A}\vec{\mathbb{c}} = \binom{1}{2} \Rightarrow \vec{\mathbb{c}} = \binom{1}{2}$.

$\vec{x} = e^{tA}\binom{1}{2} = \cdots$

A different way to find e^{tA}

e^{tA} is a fundamental matrix.

Assume $\Phi(t)$ is also a fundamental matrix.

Then, the first column of $\Phi(t) = e^{tA}\vec{\mathbb{c}_1}$ and the second column of $\Phi(t) = e^{tA}\vec{\mathbb{c}_2}$

$\Phi(t) = e^{tA}C, C = (\vec{\mathbb{c}_1}, ..., \vec{\mathbb{c}_n})$

$\Phi(0) = C \Rightarrow \Phi(t) = e^{tA}\Phi(0) \Rightarrow e^{tA} = \Phi(t)(\Phi(0))^{-1}$

Example

Solve the IVP:

$$\begin{cases} x' = 2x + y \\ y' = x + 2y \\ x(0) = 1 \\ y(0) = -1 \end{cases}$$

Solution

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \overset{\text{characteristic polynomial}}{\Rightarrow} p(z) = (2-z)^2 - 1$$

$$p(z) = 0 \Rightarrow z \in \{1, 3\} \therefore p(D)[y] = 0 \Rightarrow y = \mathbb{c}_1 e^t + \mathbb{c}_2 e^{3t}$$

Find the NFSoS at $t_0 = 0$:

$$\begin{cases} y(0) = y_0 \\ y'(0) = y_1 \end{cases} \Rightarrow \begin{cases} \mathbb{c}_1 + \mathbb{c}_2 = y_0 \\ \mathbb{c}_1 + 3\mathbb{c}_2 = y_1 \end{cases} \Rightarrow \begin{cases} \mathbb{c}_1 = \frac{3y_0 - y_1}{2} \\ \mathbb{c}_2 = \frac{y_1 - y_0}{2} \end{cases}$$

$$\Rightarrow y = \frac{3y_0 - y_1}{2}e^t + \frac{y_1 - y_0}{2}e^{3t} = y_0 \underbrace{\left(\frac{3e^t - e^{3t}}{2}\right)}_{N_0} + y_1 \underbrace{\left(\frac{e^{3t} - e^t}{2}\right)}_{N_1}$$

$$e^{tA} = \left(\frac{3e^t - e^{3t}}{2}\right)I + \left(\frac{e^{3t} - e^t}{2}\right)A$$

$$\vdots$$

$$\text{The solution is } \vec{x}(t) = e^{tA}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Example

Evaluate e^{tA} where

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

Solution

The characteristic polynomial is $p(z) = \det(A - zI) = (1-z)^2(2-z)$.

The general solution to $P(D)[y] = 0$ is $y = \mathbb{c}_1 e^t + \mathbb{c}_2 te^t + \mathbb{c}_3 e^{2t}$.

$$\begin{cases} y(0) = y_0 \\ y'(0) = y_1 \\ y''(0) = y_2 \end{cases} \Rightarrow \begin{cases} \mathbb{c}_1 + \mathbb{c}_3 = y_0 \\ \mathbb{c}_1 + \mathbb{c}_2 + 2\mathbb{c}_3 = y_1 \\ \mathbb{c}_1 + 2\mathbb{c}_2 + 4\mathbb{c}_3 = y_2 \end{cases} \Rightarrow \begin{cases} \mathbb{c}_1 = 2y_1 - y_2 \\ \mathbb{c}_2 = 3y_1 - 2y_0 - y_2 \\ \mathbb{c}_3 = y_2 - 2y_1 + y_0 \end{cases}$$

$$\Rightarrow y = (2y_1 - y_2)e^t + (3y_1 - 2y_0 - y_2)te^t + (y_2 - 2y_1 + y_0)e^{2t}$$

$$\Rightarrow y = y_0\underbrace{\left(-2te^t + e^{2t}\right)}_{N_0} + y_1\underbrace{\left(2e^t + 3te^t - 2e^{2t}\right)}_{N_1} + y_2\underbrace{\left(-e^t - te^t + e^{2t}\right)}_{N_3}$$

$$\Rightarrow e^{tA} = N_0(t)I + N_1(t)A + N_2(t)A^2$$

Example

$$\begin{cases} \frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = A\vec{x} \\ \vec{x}(0) = \vec{x_0} \end{cases}$$

Solution

The general solution to $\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = A\vec{x}$ is $\vec{x} = e^{tA}\vec{\mathbb{c}}$, where $\vec{\mathbb{c}} \in \mathbb{R}$.

$$t_0 \Rightarrow \vec{x}(0) = e^{0A}\vec{\mathbb{c}} = \vec{\mathbb{c}} \therefore \boxed{\vec{x}(t) = e^{tA}\vec{x_0}}$$

Example

Prove if $AB = BA$, then $e^A e^B = e^{A+B}$.

Scratch

$$e^{tA}e^{tB} = e^{t(A+B)}$$

$$@t = 0, \begin{cases} e^{tA}e^{tB} = I \cdot I = I \\ e^{t(A+B)} = I \end{cases}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{t(A+B)}\right) = (A+B)e^{t(A+B)}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{tA}e^{tB}\right) = Ae^{tA}e^{tB} + e^{tA}Be^{tB}$$

$$= \left(Ae^{tA} + e^{tA}B\right)e^{tB}$$

So, for $(A+B)e^{t(A+B)} = \left(Ae^{tA} + e^{tA}B\right)e^{tB}$, $e^{tA}B = Be^{tA}$.

$$@t = 0 : \begin{cases} e^{tA}B = B \\ Be^{tA} = B \end{cases}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{tA}B\right) = Ae^{tA}B \Rightarrow \frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = A\vec{x}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(Be^{tA}\right) = BAe^{tA} = ABe^{tA} \Rightarrow \frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = A\vec{x}$$

Solution

Consider the system $\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = A\vec{x}$ called $*$.(

$$\frac{\mathrm{d}}{\mathrm{d}t(Be^{tA})} = BAe^{tA} = A\left(Be^{tA}\right) \Rightarrow \text{columns of } Be^{tA} \text{ satisfy } *$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{tA}B\right) = A\left(e^{tA}B\right) \Rightarrow \text{columns of } e^{tA}B \text{ satisfy } *$$

$$e^{tA}B\Big|_{t=0} = e^{0A}B = IB = B$$

$$Be^{tA}\Big|_{t=0} = Be^{0A} = BI = B$$

$$\Rightarrow e^{tA}B\Big|_{t=0} = Be^{tA}\Big|_{t=0}$$

Thus, columns of $e^{tA}B$ and Be^{tA} satisfy the same IVP. Thus, $e^{tA}B = Be^{tA}$ by the existence and uniqueness theorem. Call that result 1.

Consider the system $\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = (A+B)\vec{x}$, called $**$.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{tA}e^{tB}\right) = Ae^{tA}e^{tB} + e^{tA}Be^{tB} \overset{\text{by 1}}{=} Ae^{tA}e^{tB} + Be^{tA}e^{tB} = (A+B)\left(e^{tA}e^{tB}\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{t(A+B)}\right) = (A+B)e^{t(A+B)}$$

Thus, columns of $e^{tA}e^{tB}$ and $e^{t(A+B)}$ satisfy $**$.

$$e^{tA}e^{tB}\Big| = e^{0A}e^{0B} = I \cdot I = I$$

$$e^{t(A+B)}\Big| = e^{0(A+B)} = I$$

Thus, by the existence and uniqueness theorem, $e^{tA}e^{tB} = e^{t(A+B)}$

Eigenpair Method

We know that e^{aT} is a solution to $y' = ay$.

This raises the question, for $\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = A\vec{x}$, is $\vec{x} = e^{ta}\vec{v}$ a solution?

$$ae^{ta}\vec{v} = Ae^{ta}\vec{v}$$
$$a\vec{v} = A\vec{v} \Rightarrow (a, \vec{v}) \text{ is an eigenpair of } A$$

Suppose $(\lambda_1, \vec{v}_1), ..., (\lambda_m, \overrightarrow{v_m})$ are eigenpairs of A such that $\vec{v}_1, \overrightarrow{v_m}$ are linearly independent. Then $e^{\lambda_1 t}\vec{v}_1, ..., e^{\lambda_m t}, \overrightarrow{v_m}$ are linearly independent over $\mathbb{C}$ (complex) solutions of $\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = A\vec{x}$.

Reason:

Suppose $\sum_{j=1}^{m} \mathbb{c}_j e^{\lambda_j t}\vec{v}_j = 0$ for all $t \in \mathbb{R}$ and some $\mathbb{c}_1, ..., \mathbb{c}_m \in \mathbb{C}$. $\sum_{j=1}^{m} \mathbb{c}_j \vec{v}_j = 0 \Rightarrow \mathbb{c}_1 = ... = \mathbb{c}_m = 0$

Theorem for eigenpair method

$$\vec{x}(t) = e^{\lambda t}\vec{v}$$

is a solution to $\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = A\vec{x}$ iff $(\lambda, \vec{v})$ is an eigenpair for A.

Example

Solve by eigenpair method:

$$\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}\vec{x}$$

Solution

Characteristic polynomial of A

$$\det\begin{pmatrix} 1-z & 2 \\ 4 & 3-z \end{pmatrix} = z^2 - 4z + 3 - 8 = (z+1)(z-5) \Rightarrow \lambda_1 = -1, \lambda_2 = 5$$

$$\text{For } \lambda = -1 \Rightarrow \begin{pmatrix} 2 & 2 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \vec{0}$$
$$\text{For } \lambda = 5 \Rightarrow \begin{pmatrix} -4 & 2 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \vec{0}$$

$\left(-1, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right) \wedge \left(5, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right)$ are eigenpairs corresponding to linearly independent eigenvectors.

Therefore, the general solution is

$$\mathbb{c}_1 e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \mathbb{c}_2 e^{5t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Example

Solve using the eigenpair method.

$$\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}\vec{x}$$

Solution

$$p(z)=(1-z)(3-z)+2=z^2-4z+5=(z-2)^2+1\to z=2\pm i.$$

$$\text{for } z=2+i: \qquad \begin{pmatrix}-1-i & 2\\ -1 & 1-i\end{pmatrix}\begin{pmatrix}1-i\\ 1\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\left(2+i,\begin{pmatrix}1-i\\ 1\end{pmatrix}\right) \text{ is an eigenpair}$$

$$\Rightarrow \vec{x}(t)=e^{(2+i)t}\begin{pmatrix}1-i\\ 1\end{pmatrix}=e^{2t}(\cos(t)+i\sin(t))\begin{pmatrix}1-i\\ 1\end{pmatrix}$$

$$=e^{2t}\begin{pmatrix}\cos(t)+\sin(t)+i(-\cos(t)+\sin(t))\\ \cos(t)+i\sin(t)\end{pmatrix}$$

Two solutions to $\frac{\mathrm{d}\vec{x}}{\mathrm{d}t}=A\begin{pmatrix}1 & 2\\ -1 & 3\end{pmatrix}$ are:

$$\begin{pmatrix}e^{2t}(\cos(t)+\sin(t))\\ e^{2t}\cos(t)\end{pmatrix},\begin{pmatrix}e^{2t}(\sin(t)-\cos(t))\\ e^{2t}\sin(t)\end{pmatrix}$$

The general solution is:

$$\mathbb{c}_1\begin{pmatrix}e^{2t}(\cos(t)+\sin(t))\\ e^{2t}\cos(t)\end{pmatrix}+\mathbb{c}_2\begin{pmatrix}e^{2t}(\sin(t)-\cos(t))\\ e^{2t}\sin(t)\end{pmatrix}$$

Example

Find e^{tA} for every $A\in M_2(\mathbb{R})$. Use that to find the general solution $\frac{\mathrm{d}\vec{x}}{\mathrm{d}t}=A\vec{x}$

Solution

Let

$$A=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$$

Then:

$$p(z)=(a-z)(d-z)-bc=z^2-(a+d)z+ad-bc=z^2-(\operatorname{tr}A)z+\det A=(z-\lambda_1)(z-\lambda_2)$$

Case I: $\lambda_1\neq\lambda_2$, $\lambda_1,\lambda_2\in\mathbb{R}$

The general solution to $P(D)[y]=0$ is $y=\mathbb{c}_1e^{\lambda_1 t}+\mathbb{c}_2e^{\lambda_2 t}$

$$\begin{cases}\mathbb{c}_1+\mathbb{c}_2=y_0\\ \lambda_1\mathbb{c}_1+\lambda_2\mathbb{c}_2=y_1\end{cases}\Rightarrow\begin{cases}\mathbb{c}_1=\frac{y_1-\lambda_2y_0}{\lambda_1-\lambda_2}\\ \mathbb{c}_2=\frac{\lambda_1y_0-y_1}{\lambda_1-\lambda_2}\end{cases}$$

Therefore:

$$\begin{aligned} y &= \mathbb{c}_1 e^{\lambda_1 t} + \mathbb{c}_2 e^{\lambda_2 t} \\ &= \frac{y_1 - \lambda_2 y_0}{\lambda_1 - \lambda_2} e^{\lambda_1 t} + \frac{\lambda_1 y_0 - y_1}{\lambda_1 - \lambda_2} e^{\lambda_2 t} \\ &= y_0 \left(\frac{\lambda_1 e^{\lambda_1 t} - \lambda_2 e^{\lambda_1 t}}{\lambda_1 - \lambda_2} \right) + y_1 \left(\frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} \right) \end{aligned}$$

Therefore:

$$\left(\frac{\lambda_1 e^{\lambda_1 t} - \lambda_2 e^{\lambda_1 t}}{\lambda_1 - \lambda_2} \right) I + \left(\frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} \right) A$$

Alternatively, we can use the eigenpair method.

$(A - \lambda_1 I)(A - \lambda_2 I) = 0 \Rightarrow$ a nonzero column of $A - \lambda_2 I$ is an eigenvector for λ_1

Use this to find a fundamental matrix, $X(t)$, and then use $e^{tA} = X(t)[X(0)]^{-1}$

Case II: $\lambda_1 = \lambda_2 \Rightarrow \lambda_1, \lambda_2 \in \mathbb{R}$

NFSOS:

$$y = \mathbb{c}_1 e^{\lambda_1 t} + \mathbb{c}_2 t e^{\lambda_2 t}$$

$$\begin{cases} \mathbb{c}_1 = y_0 \\ \lambda_1 \mathbb{c}_1 + \mathbb{c}_2 = y_1 \end{cases} \Rightarrow \begin{cases} \mathbb{c}_1 = y_0 \\ \mathbb{c}_2 = y_1 - \lambda_1 y_0 \end{cases}$$

$$\begin{aligned} y &= y_0 e^{\lambda_1 t} + (y_1 - \lambda_1 y_0) t e^{\lambda_1 t} \\ &= y_0 \left(e^{\lambda_1 t} - \lambda_1 t e^{\lambda_1 t} \right) + y_1 t e^{\lambda_1 t} \end{aligned}$$

Therefore:

$$e^{tA} = \left(e^{\lambda_1 t} - \lambda_1 t e^{\lambda_1 t} \right) I + \left(t e^{\lambda_1 t} \right) A$$

If $A\vec{v} = \lambda \vec{v} \Rightarrow e^{tA}\vec{v} = \left(e^{\lambda_1 t} - \lambda_1 t e^{\lambda_1 t} \right) \vec{v} + \left(t e^{\lambda_1 t} \right) \lambda \vec{v} = e^{\lambda_1 t} \vec{v}$

A second linearly independent solution can be found by picking $\vec{w}$ that is not a multiple of $\vec{v}$:

$$e^{tA}\vec{w} = e^{\lambda_1 t}\vec{w} - \lambda_1 t e^{\lambda_1 t}\vec{w} + t e^{\lambda_1 t} A \vec{w}$$

(

Alternatively,

$$e^{tA} = e^{t\lambda I} e^{tA - t\lambda I} = \begin{pmatrix} e^{t\lambda_1} & 0 \\ 0 & e^{t\lambda_1} \end{pmatrix} \left(I + (t(A - \lambda_1 I)) + \underbrace{\frac{t^2 (A - \lambda_1 I)^2}{2!} + \cdots}_{\text{all zero by the Cayley-Hamilton theorem}} \right) = e^{t\lambda_1} ((1 - \lambda_1 t) I + tA)$$

Case III: $\lambda_1 = \overline{\lambda_2}$, $\lambda_1, \lambda_2 \in \mathbb{C}$

The solution is online.

Example
Suppose

$$\begin{pmatrix} e^t \\ e^t \end{pmatrix}, \begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix}$$

form a FSOS for $\vec{x}' = A\vec{x}$.

1. Find A.
2. Find e^{tA}

Solution
Substitute:

$$\begin{pmatrix} e^t \\ e^t \end{pmatrix} = A \begin{pmatrix} e^t \\ e^t \end{pmatrix}$$
$$\begin{pmatrix} 2e^{2t} \\ 4e^{2t} \end{pmatrix} = A \begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix}$$

$$\begin{pmatrix} e^t & 2e^{2t} \\ e^t & 4e^{2t} \end{pmatrix} = A \begin{pmatrix} e^t & e^{2t} \\ e^t & 2e^{2t} \end{pmatrix} \Rightarrow \cdots$$

Second method:

$$e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \text{and } e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \text{ are solutions} \Rightarrow \left(1, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right), \left(2, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right)$$
$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{-1} (PDP^{-1})$$

2. Find e^{tA}

$$X(t) = \begin{pmatrix} e^t & e^{2t} \\ e^t & 2e^{2t} \end{pmatrix}$$

Thus,

$$e^{tA} = \begin{pmatrix} e^t & e^{2t} \\ e^t & 2e^{2t} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{-1}$$

Variation of Parameters

We need to find Y_p.

Suppose $\Phi(t)$ is a fundamental matrix for $\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = A\vec{x}$

Objective: find a particular solution for $\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = A\vec{x} + \vec{f}(t)$

For example, we did the following:

$$y'' + y = 0 \land y'' + y = t^2 \tan(t) \Rightarrow y_p = u_1(t)\cos(t) + u_2(t)\sin(t)$$

Now, we will do something similar:

Let $\vec{x_p} = \Phi(t)\vec{u}(t)$

Substitute into the nonhomogenous system:

$$\underbrace{\Phi'(t)\vec{u}(t) + \Phi(t)\vec{u}'(t)}_{\frac{\mathrm{d}\vec{x}}{\mathrm{d}t}} = A\Phi(t)\vec{u}(t) + \vec{f}(t)$$

Since every column of $\Phi(t)$ is a solution to $\vec{x}' = A\vec{x}$, $\Phi'(t) = A\Phi(t)$.

$$A\Phi(t)\vec{u}(t) + \Phi(t)\vec{u}'(t) = A\Phi(t)\vec{u}(t) + \vec{f}(t)$$
$$\Phi(t)\vec{u}'(t) = \vec{f}(t) \Rightarrow \vec{u}'(t) = (\Phi(t))^{-1}\vec{f}(t)$$
$$\Rightarrow \vec{u}(t) = \int (\Phi(t))^{-1}\vec{f}(t)\mathrm{d}t$$

Suppose we'd like to solve the following:

$$\begin{cases} \frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = A\vec{x} + \vec{f}(t) \\ \vec{x}(t_0) = \vec{x_0} \end{cases}$$

We know we need

$$\Phi(t_0)\vec{u}(t_0) = \vec{x_0} \Rightarrow \vec{u}(t_0) = [\Phi(t_0)]^{-1}\vec{x_0}$$
$$\Rightarrow \vec{u}(t) = \vec{u}(t_0) + \int_{t_0}^{t} (\Phi(t))^{-1}\vec{f}(t)\mathrm{d}t$$

$$\vec{x}(t) = \Phi(t)[\Phi(t_0)]^{-1}\vec{x_0} + \Phi(t)\int_{t_0}^{t} (\Phi(s))^{-1}\vec{f}(s)\mathrm{d}s$$

$$\vec{x}(t) = e^{tA}e^{-t_0A}\vec{x_0} + \int_{t_0}^{t} e^{tA}e^{-sA}\vec{f}(s)\mathrm{d}s$$
$$= e^{(t-t_0)A}\vec{x_0} + \int_{t_0}^{t} e^{(t-s)A}\vec{f}(s)\mathrm{d}s$$

Example
Solve:

$$\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = \begin{pmatrix} 4 & 5 \\ -2 & -2 \end{pmatrix}\vec{x} + \begin{pmatrix} 4e^t\cos(t) \\ 0 \end{pmatrix}, \vec{x}(0) = 0$$

Solution

$$e^{tA} = \begin{pmatrix} e^t\cos(t) + 3e^t\sin(t) & 5e^t\sin(t) \\ -2e^t\sin(t) & e^t\cos(t) - 3e^t\sin(t) \end{pmatrix}$$

Then

$$
\begin{aligned}
\vec{x}(t) &= \vec{0} + e^{tA}\int_0^t e^{-sA}\begin{pmatrix} 4e^s\cos(s) \\ 0 \end{pmatrix}\mathrm{d}s \\
&= \cancel{e^{tA}}\int_0^t e^{-s}\begin{pmatrix} \cos(-s)+3\sin(-s) & 5\sin(-s) \\ -2\sin(-s) & \cos(-s)-3\sin(-s) \end{pmatrix}\begin{pmatrix} 4\cancel{e^s}\cos(s) \\ 0 \end{pmatrix}\mathrm{d}s \\
&= 4e^{tA}\int_0^t \begin{pmatrix} \cos^2(s) - 3\sin(s)\cos(s) \\ 2\sin(s)\cos(s) \end{pmatrix}\mathrm{d}s \\
&= \cdots
\end{aligned}
$$

Laplace Transform

Define:

$$
\mathcal{L}\left\{\begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}\right\} = \begin{pmatrix} \mathcal{L}\{x_1(t)\} \\ \vdots \\ \mathcal{L}\{x_n(t)\} \end{pmatrix}
$$

Then:

$$
\mathcal{L}\{\vec{x}'(t)\}(s) = s\mathcal{L}\{\vec{x}(t)\}(s) - \vec{x}(0)
$$

$$
\vec{x}'(t) = A\vec{x}(t) + \vec{f}(t)
$$

Suppose $\vec{x}(s) = \mathcal{L}\{\vec{x}(t)\}(s)$ and $\vec{F}(s) = \mathcal{L}\left\{\vec{f}(t)\right\}(s)$.

$$
\begin{aligned}
s\vec{x}(s) - \vec{x}(0) &= A\vec{x}(s) + \vec{F}(s) \\
\Rightarrow s\vec{x} - A\vec{x}(s) &= \vec{F}(s) + \vec{x}(0) \\
\Rightarrow (sI - A)\vec{x}(S) &= \vec{F}(s) + \vec{x}(0) \\
\Rightarrow \vec{x}(s) &= (sI - A)^{-1}(F(s) - \vec{x}(0))
\end{aligned}
$$

Example

Solve using the method of Laplace transforms:

$$
\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}\vec{x} + \begin{pmatrix} e^t \\ 0 \end{pmatrix}, \vec{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}
$$

Solution

Let $X(s) = \mathcal{L}\{\vec{x}\}(s)$.

$$
sX(S) - \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}X(s) + \begin{pmatrix} \frac{1}{s-1} \\ 0 \end{pmatrix}
$$

Move $X(s)$ terms to one side:

$$sIX(S) - \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} X(s) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{1}{s-1} \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} s-1 & -4 \\ -1 & s-1 \end{pmatrix} X(s) = \begin{pmatrix} \frac{1}{s-1} \\ 1 \end{pmatrix}$$

Left-multiply the inverse:

$$\begin{aligned} X(s) &= \begin{pmatrix} s-1 & -4 \\ -1 & s-1 \end{pmatrix}^{-1} \begin{pmatrix} \frac{1}{s-1} \\ 1 \end{pmatrix} \\ &= \frac{1}{(s-1)^2 - 4} \begin{pmatrix} s-1 & 4 \\ 1 & s-1 \end{pmatrix} \begin{pmatrix} \frac{1}{s-1} \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{5}{(s-3)(s+1)} \\ \frac{1}{(s-3)(s+1)(s-1)} + \frac{s-1}{(s-3)(s+1)} \end{pmatrix} \end{aligned}$$

The inverse Laplace can be found via partial fractions, and results in:

$$\vec{x}(t) = \mathcal{L}^{-1}\{X(s)\} = \begin{pmatrix} \frac{5(e^{4t}-1)}{4e^t} \\ \frac{5-2e^{2t}+5e^{4t}}{8e^t} \end{pmatrix}$$

Chapter 13: Qualitative Theory of Differential Equations

The main focus is on autonomous systems:

Definition of Autonomous System:

Any system of this form is autonomous:

$$\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = \vec{f}(\vec{x})$$

Stationary

A solution to a system $\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = \vec{f}(\vec{x})$ is called stationary, equilibrium, fixed point or a critical point if it is a constant function.

Semistationary

A solution to a system $\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = \vec{f}(\vec{x})$ is called semistationary if all components of $\vec{x}(t)$, except for one, are constant.

Example

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = (x-1)y \\ \frac{\mathrm{d}y}{\mathrm{d}t} = xy \end{cases}$$

Solution

Some solutions for this are

$$\begin{cases} x = 1 \\ y = e^t \end{cases}, \begin{cases} x = 1 \\ x = 0 \end{cases}$$

It also seems to be that you can't cross the same value from different starting solutions.

Example

$$\begin{cases} x' = y^2 - 1 \\ y' = xy^2 + x \end{cases}$$

Solution

Find some stationary solutions:

$$\begin{cases} 0 = y^2 - 1 \Rightarrow y = \pm 1 \\ 0 = xy^2 + x \Rightarrow 0 = x(y^2 + 1) \Rightarrow 0 = x \end{cases}$$

To find the semistationary solutions, use one of those constraints, then solve.

Example

$$\begin{cases} x' = x^2 - xy - x + y \\ y' = y(x^2 - 2x + 3) \end{cases}$$

Solution

$$\begin{cases} 0 = (x - y)(x - 1) \\ 0 = y(x^2 - 2x + 3) \end{cases} \Rightarrow \begin{cases} 0 = x(x - 1) \\ y = 0 \end{cases} \Rightarrow \begin{cases} x \in \{1, 0\} \\ y = 0 \end{cases}$$

The stationary solutions are $(0, 0), (1, 0)$

Semistationary

We need a constant x that makes $x' = (x - y)(x - 1)$ zero or a constant y that makes $y(x^2 - 2x - 3)$ zero. Therefore, $x = 1$ or $y = 0$.

$$x = 1 \Rightarrow y' = 2y \Rightarrow y = \mathbb{c}e^{2t}, \mathbb{c} \in \mathbb{R} \Rightarrow (1, \mathbb{c}e^{2t})$$

$$x' = x^2 - x \Rightarrow \cdots \Rightarrow \left(\frac{1}{1 + \mathbb{c}e^t}, 0\right)$$

Questions

There are questions one would like to ask:

1. Are there stationary solutions?
2. Are there semistationary solutions?
3. What happens to a solution if the initial value is slightly modified?
4. What happens to a solution over the long run as t gets larger?
5. Are there periodic solutions?

Orbit Equation

The orbit equation of the system

$$\begin{cases} x' = f(x, y) \\ y' = g(x, y) \end{cases}$$

is

$$g(x, y)\frac{\mathrm{d}x}{\mathrm{d}t} - f(x, y)\frac{\mathrm{d}y}{\mathrm{d}t}$$

Example

$$\begin{cases} x' = 2y \\ y' = 2x - 4x^3 \end{cases}$$

$$2y'y = (2x - 4x^3)x'$$
$$2y'y + (4x^3 + 2x)x' = 0$$
$$2y\mathrm{d}y + (4x^3 + 2x)\mathrm{d}x = 0$$

$$\frac{\partial}{\partial y}(4x^3 - 2x) = 0 = \frac{\partial}{\partial x}(2y)$$

$$\varphi_x = 4x^3 - 2x \Rightarrow \varphi = x^4 - x^2 + f(y)$$
$$\varphi_y = 2y \Rightarrow \varphi = y^2 + g(x)$$
$$\therefore \varphi = x^2 - x^2 + y^2 + \mathbb{c}$$
$$\therefore x^4 - x^2 + y^2 = \mathbb{c}$$

Stability

A solution $\vec{\varphi}_0(t)$ to $\vec{x}' = \vec{f}(\vec{x})$ is stable if

$$\|\vec{\varphi}(0) - \vec{\varphi}_0(0)\| < \delta \Rightarrow \|\vec{\varphi}(t) - \vec{\varphi}_0(t)\| < \varepsilon \qquad \forall t, \forall \varepsilon > 0, \exists \delta > 0$$

For every solution $\vec{\varphi}(t)$ to $\vec{x}' = \vec{f}(\vec{x})$ and every future t. We assume all solutions are defined over the largest possible open interval.

Notes

All examples today will solutions to be a linear system with constant coefficients:

$$\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = A\vec{x}$$

Vibe based definition:

$\overrightarrow{\varphi_0}(t)$, a solution to $\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = \vec{f}(\vec{x})$, is called stable if every solution that starts near $\overrightarrow{\varphi_0}(t)$ stays near $\overrightarrow{\varphi_0}(t)$ forever.

Example

$$A = \begin{pmatrix} -3 & 1 \\ -2 & 0 \end{pmatrix}$$

Show whether the $\vec{0}$ solution to $\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = A\vec{x}$ is stable.

Solution

The general solution is

$$\vec{\varphi}(t) = \mathbb{c}_1 e^{-2t}\begin{pmatrix}1\\1\end{pmatrix} + \mathbb{c}_2 e^{-t}\begin{pmatrix}1\\2\end{pmatrix}$$

Scratch

$$\forall \varepsilon > 0, \exists \delta > 0 \left\| \mathbb{c}_1 \begin{pmatrix}1\\1\end{pmatrix} + \mathbb{c}_2 \begin{pmatrix}1\\2\end{pmatrix} - \vec{0} \right\| \leq \delta \Rightarrow \left\| \mathbb{c}_1 e^{-2t}\begin{pmatrix}1\\1\end{pmatrix} + \mathbb{c}_2 e^{-t}\begin{pmatrix}1\\2\end{pmatrix} - \vec{0} \right\| \leq \varepsilon$$

Our objective is to find δ in terms of ε.

$$\left\| \mathbb{c}_1 \begin{pmatrix}1\\1\end{pmatrix} + \mathbb{c}_2 \begin{pmatrix}1\\2\end{pmatrix} \right\| < \delta \Rightarrow \left\| \begin{pmatrix}\mathbb{c}_1 + \mathbb{c}_2\\ \mathbb{c}_1 + 2\mathbb{c}_2\end{pmatrix} \right\| < \delta \Rightarrow \sqrt{(\mathbb{c}_1 + \mathbb{c}_2)^2 + (\mathbb{c}_1 + 2\mathbb{c}_2)^2} < \delta$$

$$\Rightarrow \begin{cases} |\mathbb{c}_1 + \mathbb{c}_2| < \delta \\ |\mathbb{c}_1 + 2\mathbb{c}_2| < \delta \end{cases} \Rightarrow \begin{cases} |\mathbb{c}_1| = |2(\mathbb{c}_1 + \mathbb{c}_2) - (\mathbb{c}_1 + 2\mathbb{c}_2)| \leq 2|\mathbb{c}_1 + \mathbb{c}_2| + |\mathbb{c}_1 + 2\mathbb{c}_2| < 3\delta \\ |\mathbb{c}_2| = |(\mathbb{c}_1 + 2\mathbb{c}_2) - (\mathbb{c}_1 + \mathbb{c}_2)| \leq |\mathbb{c}_1 + 2\mathbb{c}_2| + |\mathbb{c}_1 + \mathbb{c}_2| < 2\delta \end{cases}$$

$$\left\| \mathbb{c}_1 + e^{-2t}\begin{pmatrix}1\\1\end{pmatrix} + \mathbb{c}_2 e^{-t}\begin{pmatrix}1\\2\end{pmatrix} \right\| \leq |\mathbb{c}_1| e^{-2t} \left\| \begin{pmatrix}1\\1\end{pmatrix} \right\| + |\mathbb{c}_2| e^{-t} \left\| \begin{pmatrix}1\\2\end{pmatrix} \right\| < (3\delta)(1)\left(\sqrt{2}\right) + (2\delta)(1)\left(\sqrt{5}\right) = \left(3\sqrt{2} + 2\sqrt{5}\right)\delta < \varepsilon$$

Therefore, we will select

$$\delta = \frac{\varepsilon}{3\sqrt{2} + 2\sqrt{5}}$$

Actual Solution

Let $\varepsilon > 0$. Set $\delta = \dfrac{\varepsilon}{100}$. Note that the general solution is

$$\vec{x}(t) = \mathbb{c}_1 e^{-2t}\begin{pmatrix}1\\1\end{pmatrix} + \mathbb{c}_2 e^{-t}\begin{pmatrix}1\\2\end{pmatrix}$$

$$\left\| \vec{x}(0) - \vec{0} \right\| < \delta \Rightarrow \left\| \mathbb{c}_1 \begin{pmatrix}1\\1\end{pmatrix} + \mathbb{c}_2 \begin{pmatrix}1\\2\end{pmatrix} \right\| < \delta \Rightarrow \sqrt{(\mathbb{c}_1 + \mathbb{c}_2)^2 + (\mathbb{c}_1 + 2\mathbb{c}_2)^2} < \delta \Rightarrow |\mathbb{c}_1 + \mathbb{c}_2| < \delta \wedge |\mathbb{c}_1 + 2\mathbb{c}_2| < \delta$$

$$\Rightarrow |\mathbb{c}_1| = |2(\mathbb{c}_1 + \mathbb{c}_2) - (\mathbb{c}_1 + 2\mathbb{c}_2)| \leq 2|\mathbb{c}_1 + \mathbb{c}_2| + |\mathbb{c}_1 + 2\mathbb{c}_2| < 3\delta$$

$$|\mathbb{c}_2| = |\mathbb{c}_1 + 2\mathbb{c}_2 - (\mathbb{c}_1 + \mathbb{c}_2)| \leq |\mathbb{c}_1 + 2\mathbb{c}_2| + |\mathbb{c}_1 + \mathbb{c}_2| \leq 2\delta$$

$$\text{If } t \geq 0, \text{then } \left| \vec{x}(t) - \vec{0} \right| = \left\| \mathbb{c}_1 e^{-2t} \begin{pmatrix}1\\1 + \mathbb{c}_2 e^{-t}(1,2)\end{pmatrix} \right\|$$

$$\leq |\mathbb{c}_1| e^{-2t} \left\| \begin{pmatrix}1\\1\end{pmatrix} \right\| + |\mathbb{c}_2| e^{-t} \left\| \begin{pmatrix}1\\2\end{pmatrix} \right\| < (3\delta)(1)\left(\sqrt{2}\right) + (2\delta)(1)\left(\sqrt{5}\right)$$

$$< 6\delta + 10\delta = 16\delta = \frac{16\varepsilon}{100} < \varepsilon$$

Thus, $\vec{0}$ is a stable solution.

Example

$$A = \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}$$

Show whether the $\vec{0}$ solution to $\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = A\vec{x}$ is stable.

Solution

The general solution is $\vec{x}(t) = \mathbb{c}_1 e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \mathbb{c}_2 e^{-t}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

We claim $\vec{0}$ is unstable. On the contrary, assume $\vec{0}$ is stable. Set $\varepsilon = 1$ in the definition of stability:

$$\exists \delta > 0 \ \left\| \mathbb{c}_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \mathbb{c}_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\| \leq \delta \Rightarrow \left\| \mathbb{c}_1 e^{2t} + \mathbb{c}_2 e^{-t} \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\| < 1$$

Set $\mathbb{c}_1 = \frac{\delta}{2}, \mathbb{c}_2 = 0$

$$\left\| \frac{\delta}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (0) \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\| = \frac{\delta}{2}\sqrt{2} < \delta.$$

Thus, we must have $\left\| \frac{\delta}{2} e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\| \leq 1$ for all $t \geq 0$

$$\Rightarrow \frac{\delta\sqrt{2}}{2} e^{2t} < 1 \Rightarrow \lim_{t \to \infty} \frac{\delta\sqrt{2}}{2} e^{2t} < 1 \Rightarrow \infty < 1 \text{ a contradiction}$$

Example

$$A = \begin{pmatrix} -2 & 1 \\ -4 & 2 \end{pmatrix}$$

Show whether the $\vec{0}$ solution to $\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = A\vec{x}$ is stable.

Solution

$$\vec{x}(t) = \mathbb{c}_1 \begin{pmatrix} 1 - 2t \\ -4t \end{pmatrix} + \mathbb{c}_2 \begin{pmatrix} t \\ 2t + 1 \end{pmatrix}$$

$$\forall \varepsilon > 0 \exists \delta > 0 \qquad \left\| \mathbb{c}_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mathbb{c}_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \vec{0} \right\| < \delta \Rightarrow \left\| \mathbb{c}_1 \begin{pmatrix} 1 - 2t \\ -4t \end{pmatrix} + \mathbb{c}_2 \begin{pmatrix} t \\ 2t + 1 \end{pmatrix} \right\| < \varepsilon$$

We claim that $\vec{0}$ is unstable. On the contrary, assume $\vec{0}$ is stable.

Let $\varepsilon = 1$ in the definition of a stable solution:

$$\exists \delta > 0 \qquad \left\| \mathbb{c}_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mathbb{c}_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \vec{0} \right\| < \delta \Rightarrow \left\| \mathbb{c}_1 \begin{pmatrix} 1 - 2t \\ -4t \end{pmatrix} + \mathbb{c}_2 \begin{pmatrix} t \\ 2t + 1 \end{pmatrix} \right\| < 1$$

Choose $\mathbb{c}_1 = \frac{\delta}{2}$, $\mathbb{c}_2 = 0$:

$$\left\| \frac{\delta}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + (0) \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \vec{0} \right\| = \frac{\delta}{2} < \delta$$

Thus $\forall t \geq 0$:

$$\left\| \frac{\delta}{2} \begin{pmatrix} 1 - 2t \\ -4t \end{pmatrix} \right\| < 1 \Rightarrow \frac{\delta}{2}\sqrt{(1 - 2t)^2 + 16t^2} < 1 \Rightarrow \frac{\delta}{2}\sqrt{16t^2} < 1 \Rightarrow 2\delta|t| < 1 \Rightarrow \lim_{t \to \infty} 2\delta|t| < 1 = \infty < 1 \text{ a contradiction}$$

Example

$$A = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$$

Solution

$$\vec{x}(t) = e^{2it}\begin{pmatrix} i \\ -1 \end{pmatrix} = \begin{pmatrix} -i\cos(2t) - \sin(2t) \\ -\cos(2t) - i\sin(2t) \end{pmatrix}$$

$$\therefore \vec{x}(t) = \mathbb{c}_1\begin{pmatrix} \sin(2t) \\ \cos(2t) \end{pmatrix} + \mathbb{c}_2\begin{pmatrix} \cos(2t) \\ -\sin(2t) \end{pmatrix}$$

Scratch

$$\left\|\mathbb{c}_1\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \mathbb{c}_2\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\| < \delta \Rightarrow \sqrt{\mathbb{c}_1^2 + \mathbb{c}_2^2} < \delta \Rightarrow |\mathbb{c}_1| < \delta, |\mathbb{c}_2| < \delta$$

$$\Rightarrow |\mathbb{c}_1|\left\|\begin{pmatrix} \sin(2t) \\ \cos(2t) \end{pmatrix}\right\| + |\mathbb{c}_2|\left\|\begin{pmatrix} \cos(2t) \\ -\sin(2t) \end{pmatrix}\right\| = |\mathbb{c}_1| + |\mathbb{c}_2| < 2\delta$$

Solution

We will show that $\vec{0}$ is stable. Let $\varepsilon > 0$ and set $\delta = \frac{\varepsilon}{2}$

If

$$\left\|\mathbb{c}_1\begin{pmatrix} \sin(0) \\ \cos(0) \end{pmatrix} + \mathbb{c}_2\begin{pmatrix} \cos(0) \\ -\sin(0) \end{pmatrix}\right\| \leq \frac{\varepsilon}{2} \Rightarrow \left\|\begin{pmatrix} 0 \\ \mathbb{c}_1 \end{pmatrix} + \begin{pmatrix} \mathbb{c}_2 \\ 0 \end{pmatrix}\right\| < \frac{\varepsilon}{2} \Rightarrow \sqrt{\mathbb{c}_1^2 + \mathbb{c}_2^2} < \frac{\varepsilon}{2} \Rightarrow |\mathbb{c}_1| < \frac{\varepsilon}{2}, |\mathbb{c}_2| < \frac{\varepsilon}{2}$$

Therefore,

$$\left\|\mathbb{c}_1\begin{pmatrix} \sin(2t) \\ \cos(2t) \end{pmatrix} + \mathbb{c}_2\begin{pmatrix} \cos(2t) \\ -\sin(2t) \end{pmatrix}\right\| \leq |\mathbb{c}_1|\left\|\begin{pmatrix} \sin(2t) \\ \cos(2t) \end{pmatrix}\right\| + |\mathbb{c}_2|\left\|\begin{pmatrix} \cos(2t) \\ -\sin(2t) \end{pmatrix}\right\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Thus, $\vec{0}$ is stable.

Asymptotically stable

A solution $\overrightarrow{\varphi_0}(t)$ is asymptotically stable if it is stable and every $\vec{\varphi}$ close to $\overrightarrow{\varphi_0}$ converges to $\overrightarrow{\varphi_0}$:

$$\exists\delta > 0 \|\vec{\varphi}(0) - (\varphi_0)(0)\| < \delta \Rightarrow \vec{\varphi}(t) \to \overrightarrow{\varphi_0}(t) \text{ as } t \text{ gets as large as possible}$$

Vibe-based Stability Examples

If you have the system $\vec{x}' = A\vec{x}$, what is the stability of $\vec{0}$?

Examples of $\vec{x}(t)$:

$$\mathbb{c}_1 e^{2t}\vec{v}_1 + \mathbb{c}_2 e^{3t}\vec{v}_2 \to \text{unstable}$$
$$\mathbb{c}_1 e^{2t}\vec{v}_1 + \mathbb{c}_2 e^{-3t}\vec{v}_2 \to \text{unstable}$$
$$\mathbb{c}_1 e^{-2t}\vec{v}_1 + \mathbb{c}_2 e^{-3t}\vec{v}_2 \to \text{asymptotically stable}$$
$$\mathbb{c}_1 \vec{v}_1 + \mathbb{c}_2 \vec{v}_2 \to \text{stable}$$
$$\mathbb{c}_1 \vec{v}_1 + \mathbb{c}_2 (2t+5)\vec{v}_2 \to \text{unstable}$$
$$\mathbb{c}_1 e^{2t}\cos(t)\vec{v}_1 + \mathbb{c}_2 e^{2t}\sin(t)\vec{v}_2 \to \text{unstable}$$
$$\mathbb{c}_1 e^{-t}\vec{v}_1 + \mathbb{c}_2 e^{-t}(2t+3)\vec{v}_3 \to \text{asymptotically stable}$$
$$\mathbb{c}_1 e^{-2t}\cos(3t)\vec{v}_1 + \mathbb{c}_2 e^{-2t}\sin(3t) \to \text{asymptotically stable}$$
$$\mathbb{c}_1 \cos(3t)\vec{v}_1 + \mathbb{c}_2 \sin(3t)\vec{v}_2 \to \text{stable}$$

$$\lambda = a + bi \Rightarrow e^{at}\cos(bt) + e^{at}\sin(bt)$$
$$\text{Results in a stable solution if } a < 0$$

If you have repeated roots of 0 with linearly independent eigenvectors:

$$\vec{x}(t) = \mathbb{c}_1 \vec{v}_1 + \mathbb{c}_2 \vec{v}_2 \to \text{stable}$$

There are different types of stability. $\mathbb{c}_1 \cos(3t)\vec{v}_1 + \mathbb{c}_2 \sin(3t)\vec{v}_2$ is stable by being bounded, but it does not reach zero. The other example is for example, $\mathbb{c}_1 e^{-2t}\cos(3t)\vec{v}_1 + \mathbb{c}_2 e^{-2t}\sin(3t)$ where it goes to zero. This is called asymptotically stable.

Stability of the solutions to $\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = A\vec{x}$

Consider the system $\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = A\vec{x}$.

- All solutions are asymptotically stable if all eigenvalues of A have negative real parts.
- All solutions are unstable if at least one eigenvalue of A has a positive real part.
- Suppose all eigenvalues of A have nonpositive real parts. Let $\lambda_1, ..., \lambda_k$, with $k \geq 1$, be all distinct eigenvalues of A whose real parts are zero. Call the multiplicity of λ_j as a root of the characteristic polynomial of A m_j.
  - If A has m_j linearly independent eigenvectors corresponding to λ_j for every j, then every solution to the system is stable but not asymptotically stable.
  - Otherwise, every solution is unstable.

Example

Take the following eigenvectors:

$$0, 0, i, i, -i, -i, -3, -4, -4, -4$$

For stability, you need $\vec{v}_1, \vec{v}_2$ as linearly independent eigenvectors of 0, and $\overrightarrow{w_1}, \overrightarrow{w_2}$ linearly independent eigenvectors of i and $\overline{\overrightarrow{w_1}}, \overline{\overrightarrow{w_2}}$ linearly independent for $-i$.

Theorem on Stability in nonlinear systems near stationary solutions

Suppose $\vec{x}_0$ is a stationary solution to the system $\vec{x}' = f(x)$. Let A be the Jacobian matrix at x_0.

- If all eigenvalues of A have negative real parts, then $\vec{x}_0$ is an asymptotically stable solution to the system.

- If at least one eigenvalue of A has a positive real part, then $\vec{x}_0$is an unstable solution to the system.
- If neither is true, then $\vec{x}_9$ could be stable, unstable or asymptotically stable.

Getting the vibe of the answer

Approximate the system with a linear system and use the stability of the linear system to understand the stability of solutions to a nonlinear system.

Pick (x_0, y_0).

$$\begin{cases} x' = f(x,y) \approx f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) \\ y' = g(x,y) \approx g(x_0,y_0) + g_x(x_0,y_0)(x-x_0) + g_y(x_0,y_0)(y-y_0) \end{cases}$$

If (x_0, y_0) is a stationary solution, then $f(x_0,y_0) = g(x_0,y_0) = 0$

This results in:

$$\begin{cases} x' = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) \\ y' = g_x(x_0,y_0)(x-x_0) + g_y(x_0,y_0)(y-y_0) \end{cases}$$

Let $\tilde{x} = x - x_0$, $\tilde{y} = y - y_0$.

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \underbrace{\begin{pmatrix} f_x(x_0,y_0) & f_y(x_0,y_0) \\ g_x(x_0,y_0) & g_y(x_0,y_0) \end{pmatrix}}_{A} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}$$

This is the linearization near the stationary solution (x_0, y_0).

If you take that approximation resulting in eigenvalues of $\{-2, -3\}$, it is asymptotically stable around the stationary solution.

If you have that approximation resulting in eigenvalues of $\{2, -3\}$, it is unstable around the stationary solution.

If you have zeros, it's unclear.

Example

$$\begin{cases} x' = 1 - xy \\ y' = x - y^3 \end{cases}$$

Solution

Stationary solutions satisfy

$$\begin{cases} 1 - xy = 0 \\ x - y^3 = 0 \end{cases} \to \begin{cases} 1 - xy = 0 \\ x = y^3 \end{cases} \to \begin{cases} 1 = y^4 \\ x = y^3 \end{cases} \to \begin{cases} x = \pm 1 \\ y = \pm 1 \end{cases}$$

The Jacobian matrix is:

$$\begin{pmatrix} -y & -x \\ 1 & -3y^2 \end{pmatrix}$$

Evaluate at (1,1):

$$\begin{pmatrix} -1 & 1 \\ 1 & -3 \end{pmatrix} \to z^2 + 4z + 4 = (z+2)^2 \Rightarrow \lambda = (-2, -2)$$

By a theorem, $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an asymptotically stable solution.

Evaluate at $(-1, -1)$:

$$\begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix} \Rightarrow z^2 + 2z - 4 \to (z+1)^2 - 5 \to \lambda = -1 \pm \sqrt{5}$$

Since $\sqrt{5} - 1 > 0$, $(-1, -1)$ is an unstable solution.

Example

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = \sin(x+y) \\ \frac{\mathrm{d}y}{\mathrm{d}t} = e^x - 1 \end{cases}$$

Solution

Find stationary solutions:

$$\begin{cases} \sin(x+y) = 0 \\ e^x - 1 = 0 \end{cases} \Rightarrow \begin{cases} \sin(x+y) = 0 \\ e^x = 1 \end{cases} \Rightarrow \begin{cases} x = 0 \\ \sin(y) = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = k\pi, k \in \mathbb{Z} \end{cases}$$

Jacobian:

$$\begin{pmatrix} \cos(x+y) & \cos(x+y) \\ e^x & 0 \end{pmatrix}$$

Plug in stationary solutions:

$$\begin{pmatrix} \cos(k\pi) & \cos(k\pi) \\ 1 & 0 \end{pmatrix} \Rightarrow p(z) = z^2 - \cos(k\pi)z - \cos(k\pi) = 0$$

$$\frac{\cos(k\pi) \pm \sqrt{\cos^2(k\pi) + 4\cos(k\pi)}}{2} \Rightarrow \begin{cases} \frac{1 \pm \sqrt{5}}{2} \text{ if k is even} \\ \frac{-1 \pm \sqrt{3}i}{2} \text{ if k is odd} \end{cases}$$

If k is even, then $\mathrm{Re}\left(\frac{1+\sqrt{5}}{2}\right) > 0$ and thus $(0, k\pi)$ is unstable.

If k is odd, then $\mathrm{Re}\left(\frac{-1\pm\sqrt{3}i}{2}\right) = -\frac{1}{2} < 0$ and thus $(0, k\pi)$ is asymptotically stable.

Review

Example of an inverse Laplace

Find the inverse Laplace of $\ln\left(\frac{s^2}{s^2+1}\right)$

Scratch

$$= 2\ln(s) - \ln(s^2 + 1)$$

$$\mathcal{L}^{-1}\left\{\ln\left(\frac{s^2}{s^2+1}\right)\right\}(t) = \mathcal{L}^{-1}\{\ln(s^2) - \ln(s^2+1)\}(t) \Rightarrow -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{2}{s} - \frac{2s}{s^2+1}\right\} \Rightarrow -\frac{1}{t}(2 - 2\cos(t)) = \frac{2\cos(2t) - 2}{t}$$

Actual solution

We claim the following is piecewise continuous and of exponential order:

$$j(t) = \frac{2\cos(t) - 2}{t}$$

It is piecewise continuous because the only discontinuity is at $t = 0$.

$$\lim_{t\to 0} \frac{2\cos(t) - 2}{t} = \lim_{t\to 0} \frac{-2\sin(t)}{1} = 0$$

$$\left|\frac{2\cos(t) - 2}{t}\right| \leq M \forall t \in (0, 1], \text{by EVT since } f(t) = \begin{cases} \frac{2-2\sin(t)}{t} \text{ if } t \neq 0 \\ 0 \text{ if } t = 0 \end{cases}$$

If $t > 1$ then

$$\left|\frac{2\cos(t) - 2}{t}\right| \leq \frac{4}{1} \forall t \in (0, \infty), \left|\frac{2\cos(t) - 2}{t}\right| \leq M + 4 \leq (M + 4)e^{0t}$$

By the table:

$$\mathcal{L}\left\{t\frac{2\cos(t) - 2}{t}\right\}(s) = -\frac{\mathrm{d}}{\mathrm{d}s}\left(\mathcal{L}\left\{\frac{2\cos(t) - 2}{t}\right\}(s)\right) = \frac{2}{s} - \frac{2s}{s^2+1}$$

$$\therefore \int -\frac{\mathrm{d}}{\mathrm{d}s}\left(\mathcal{L}\left\{\frac{2\cos(t) - 2}{t}\right\}(s)\right)\mathrm{d}s = \int \frac{2}{s} - \frac{2s}{s^2+1}\mathrm{d}s$$

$$\mathcal{L}\left\{\frac{2\cos(t) - 2}{t}\right\}(s) = 2\ln(s) - \ln(s^2+1) + \mathbb{c}_1$$

$$\text{At } s \to \infty : 0 \therefore \mathbb{c}_1 = 0$$

$$\mathcal{L}\left\{\frac{2\cos(t) - 2}{t}\right\}(s) = \ln(s^2) - \ln(s^2+1)$$

$$\mathcal{L}\left\{\frac{2\cos(t) - 2}{t}\right\}(s) = \ln\left(\frac{s^2}{s^2+1}\right)$$

Example 8.6

Suppose two different linear homogeneous differential equations with constant coefficients $L_1[y] = 0$ and $L_2[y] = 0$ have a common nonzero solution $y = y(t)$, defined over R. Prove that the characteristic polynomials $p_1(z)$ and $p_2(z)$ of L_1 and L_2 have at least one common root.

Suppose the list of all distinct roots to $p_1(z)$ is:

$$z_1, z_2, ..., z_m$$

And the list of all distinct roots to $p_2(z)$ is:

$$w_1, w_2, ..., w_l$$

Assume, on the contrary, $z_{j'}s$ and $w_{j'}s$ are distinct:

$$\begin{aligned} y &= \sum_{k=1}^{s} \sum_{j=1}^{m} \mathbb{c}_{jk} t^k e^{z_j t} \text{ for some } \mathbb{c}_{jk} \in \mathbb{C} \\ &= \sum_{k=1}^{r} \sum_{j=1}^{l} \mathbb{d}_{jk} t^k e^{w_j t} \text{ for some } \mathbb{d}_{jk} \in \mathbb{C} \end{aligned}$$

$$\sum_{k=1}^{s} \sum_{j=1}^{m} \mathbb{c}_{jk} t^k e^{z_j t} - \sum_{k=1}^{r} \sum_{j=1}^{l} \mathbb{d}_{jk} t^k e^{w_j t} = 0$$

Since $z_1, ..., z_m, w_1, ..., w_l$ are distinct, $\mathbb{c}_{jk} = \mathbb{d}_{jk} = 0 \forall j, k \Rightarrow y = 0$

Chapter 14: Orbits and PHase Plane Portraits

Orbits

$$\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = \vec{f}(\vec{x}); \vec{x} : (a, b) \to \mathbb{R}^n$$

When drawing a graph of an orbit in the xy-plane, arrows indicate the direction in which t increases.

$$\begin{cases} x' = f(x, y) \\ y' = g(x, y) \end{cases}$$

forms the orbit equation $f(x, y)\mathrm{d}y - g(x, y)\mathrm{d}x = 0$.

Define

Given a system $\frac{\mathrm{d}x}{\mathrm{d}t} = f(x, y), \frac{\mathrm{d}y}{\mathrm{d}t} = g(x, y)$:

An orbit is the curve $(x(t), y(t))$ on the xy-plane, where $(x(t), y(t))$ is a solution to the system.

Phase plane portraits are formed by drawing sample orbits in the xy-plane that show the behavior of all solutions.

Once again, arrows indicate the direction in which each orbit is traversed t gets as large as possible.

Existence and Uniqueness Theorem for Autonomous Systems

$$\begin{cases} \frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = \vec{f}(\vec{x}) \\ \vec{x}(t_0) = \vec{x}_0 \end{cases}$$

Assume $\vec{f}$ is C^1 over an open subset U of $\mathbb{R}^n$ containing $\vec{x}_0$.

Then, a unique solution is defined over some interval, $(t_0 - \varepsilon, t_0 + \varepsilon)$.

Unrelated to that theorem, but if $\vec{x}(t)$ is a solution to $\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = \vec{f}(\vec{x})$, then so is $\vec{x}(t + c)$ for a constant $c \in \mathbb{R}$, as long as both t an $t + c$ are in the domain of $\vec{x}$.

Proof

Let $\vec{y}(t) = \vec{x}(t + c)$.

$$\vec{y}'(t) = \vec{x}'(t+c) = \vec{f}(\vec{x}(t+c)) = \vec{f}(\vec{y}(t))$$

Theorem on Properties of Orbits

Suppose all components of the vector field $\vec{f}(x)$ have continuous first partials, then:

1. Two distinct orbits do not intersect.
2. If $\vec{\varphi}(t)$ is a solution to $\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = \vec{f}(\vec{x})$ with $\vec{f} \in C^1$, and $\vec{\varphi}(t_0) = \vec{\varphi}(t_0 + T)$ for some t_0, T, with $T \neq 0$, them $\vec{\varphi}$ is periodic with period T.
3. The corresponding solution is periodic if an orbit lies on a closed curve containing no stationary solution.

Proof

Suppose $\vec{x}$ and $\vec{y}$ are both solutions to $\frac{\mathrm{d}\vec{z}}{\mathrm{d}t} = \vec{f}(\vec{z})$ and $\vec{x}(t_0) = \vec{y}(t_1)$. Consider the IVP:

$$\begin{cases} \frac{\mathrm{d}\vec{z}}{\mathrm{d}t} = \vec{f}(\vec{z}) \\ \vec{z}(t_0) = \vec{x}(t_0) \end{cases}$$

$\vec{x}(t)$ satisfies this IVP by the choice of $\vec{x}$. $\vec{z}(t) = \vec{y}(t + t_1 - t_0)$ satisfies the differential equation.

Further,

$$z(t_0) = \vec{y}(t_0 + t_1 - t_0) = \vec{y}(t_1) = \vec{x}(t_0)$$

.

$\vec{z}$ is another solution to the IVP. $\vec{z} = \vec{x} \Rightarrow \vec{y}(t + t_1 - t_0) = \vec{x}(t)$, as long as t is in the domain of $\vec{x}$, and $t + t_1 - t_0$ is in the domain of $\vec{y}$.

These two solutions may be offset by some time but represent the same order.

Arc Length

$$\begin{cases} x' = f(x,y) \\ y' = g(x,y) \end{cases}$$

$$\int_0^s \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2} \mathrm{d}t = \int_0^s \sqrt{f^2 + g^2} \mathrm{d}t$$

$f^2 + g^2 \neq 0$ since there is no stationary solution. Therefore,

$$\lim_{s \to \infty} \int_0^s \sqrt{f^2 + g^2} \mathrm{d}t = \infty$$

Example

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = y e^{1+x^2+y^2} \\ \frac{\mathrm{d}y}{\mathrm{d}t} = -x e^{1+x^2+y^2} \end{cases}$$

Solution

The orbit equation:

$$ye^{1+x^2+y^2}\mathrm{d}y + xe^{1+x^2+y^2}\mathrm{d}x = 0$$

$$\Rightarrow y\mathrm{d}y + x\mathrm{d}x = 0 \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t\left(\frac{1}{2}(x^2+y^2)\right)} = 0 \Rightarrow x^2 + y^2 = \mathbb{c}$$

Stationary Solutions
$(0, 0)$ is the only stationary solution. This is the solution represented by $C = 0$.

If $C > 0$, the orbit lies on the closed curve $x^2 + y^2 = C$, which contains no stationary solutions, so the solution is periodic.

If $C = 0$, then $x^2 + y^2 = 0 \Rightarrow x = y = 0$, which is also periodic.

$C < 0$ yields no curve in the reals.

Example

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = xy \\ \frac{\mathrm{d}y}{\mathrm{d}t} = e^x + y^2 \end{cases}$$

Solution
We will show the y-axis is the union of orbits. Since orbits do not intersect if a solution starts with $x > 0$, it can never get to $x = 0$ or $x \leq 0$.

Set $x = 0$:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = e^0 + y^2 = 1 + y^2 \Rightarrow \arctan(y) = t + \mathbb{c} \Rightarrow y = \tan(t + \mathbb{c}).$$

Take a solution to the above:

$$(0, \tan(t))$$

This is one orbit that covers the entire y-axis ($-\frac{\pi}{2} < t < \frac{\pi}{2}$).

Example

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = y \\ \frac{\mathrm{d}y}{\mathrm{d}t} = -2x - 4x^3 \end{cases}$$

Solutions
Stationary Solutions:

$$\begin{cases} y = 0 \\ -2x - 4x^3 = 0 \end{cases} \overset{\text{over } \mathbb{R}}{\Rightarrow} \begin{cases} x = 0 \\ y = 0 \end{cases}$$

Orbit equation, which is exact:

$$y\mathrm{d}y + (2x + 4x^3)\mathrm{d}x = 0$$

Therefore

$$\frac{y^2}{2} + x^2 + x^4 = \mathbb{c}$$

Now, to show periodicity, check that it is a closed curve, for $\mathbb{c} \geq 0$

$$\frac{y^2}{2} + x^2 + x^4 = \mathbb{c}$$
$$\Rightarrow y = \pm\sqrt{2\mathbb{c} - 2x^2 - 2x^4}$$
$$2\mathbb{c} - 2x^2 - 2x^4 \geq 0$$
$$x^2 + x^4 - \mathbb{c} \leq 0$$
$$x^4 + x^2 - \mathbb{c} = (x^2 - r)(x^2 - s)$$
$$\text{Assume } \mathbb{c} > 0:$$
$$rs = -\mathbb{c} < 0$$
$$\text{Thus } r < 0 < s \Rightarrow x^2 - r > 0$$
$$(x^2 - r)(x^2 - s) \leq 0 \Rightarrow -\sqrt{s} \leq x \leq \sqrt{s}$$

Thus, if $\mathbb{c} > 0$, the orbit lies on a closed curve.

The stationary solution is $(0, 0)$. If $(0, 0)$ is on the curve, then $\mathbb{c} = 0$. Thus, for every $\mathbb{c} > 0$, the solution is periodic.

If $\mathbb{c} = 0$, then $x = y = 0$, which is a fixed point, and therefore the solution is periodic.

Phase plane Portraits

Sample orbits of various kinds, along with arrows indicating the behaviors of the solution as t increases.

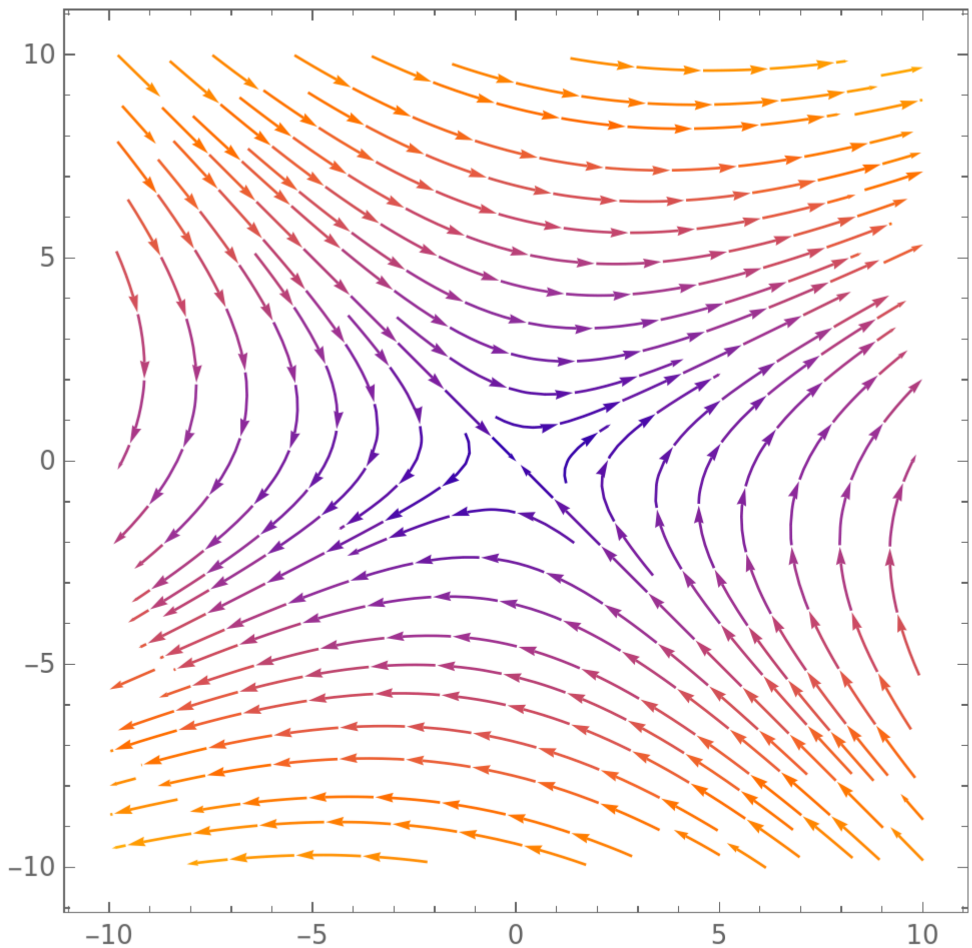
Example

$$\vec{x}(t) = \mathbb{c}_1 e^{-3t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \mathbb{c}_2 e^{3t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$A\vec{x} = \vec{0} \Rightarrow \vec{x} = \vec{0}$

Solution

$$\vec{x}(t) = \mathbb{c}_1 e^{-3t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \Rightarrow y = -x, x < 0, \mathbb{c}_1 > 0$$
$$\vec{x}(t) = \mathbb{c}_1 e^{-3t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \Rightarrow y = -x, x > 0, \mathbb{c}_1 < 0$$
$$\vec{x}(t) = \mathbb{c}_2 e^{3t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} : y = \frac{1}{2}x$$
$$\mathbb{c}_1 e^{-3t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \mathbb{c}_2 e^{3t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \overset{t \to \infty}{\approx} \mathbb{c}_2 e^{3t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$
$$\mathbb{c}_1 e^{-3t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \mathbb{c}_2 e^{3t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \overset{t \to -\infty}{\approx} \mathbb{c}_2 e^{-3t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$



This solution is called a saddle.

Example

$$\vec{x}(t) = \mathbb{c}_1 e^t \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \mathbb{c}_2 e^{2t} \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

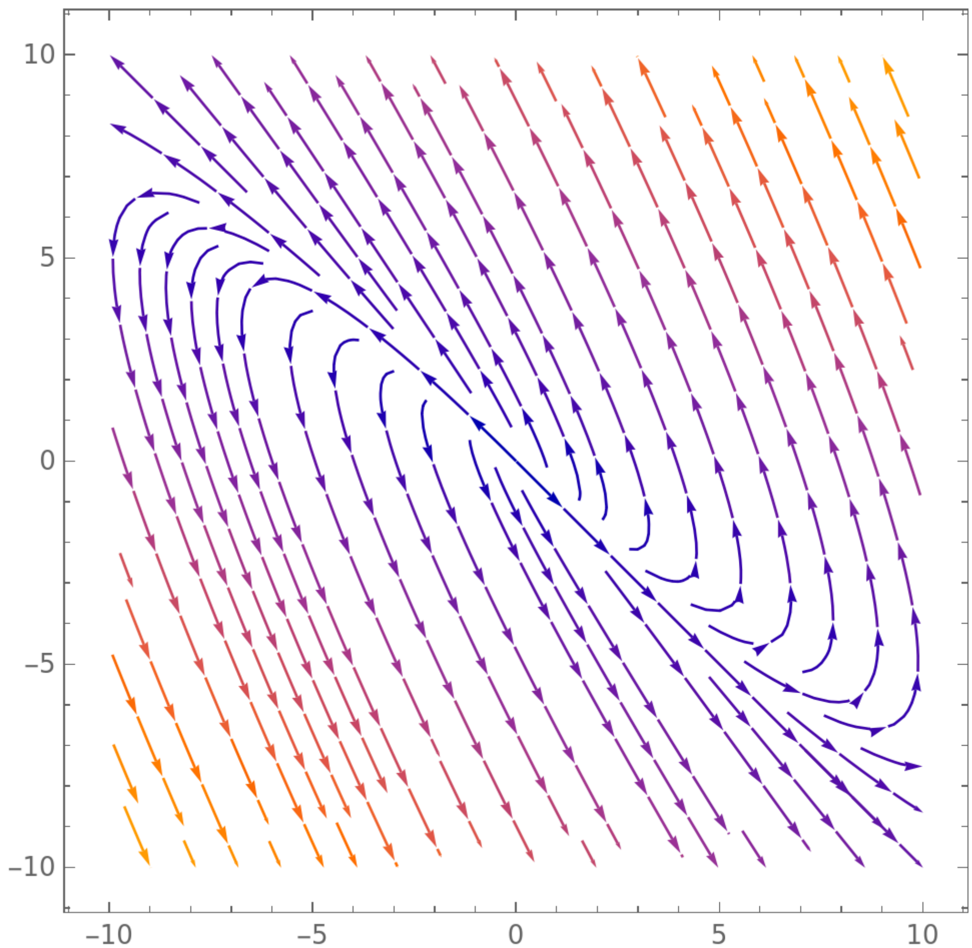
Solution

$$\mathbb{c}_1 e^t \begin{pmatrix} -1 \\ 1 \end{pmatrix} : y = -x$$

$$\mathbb{c}_2 e^{2t} \begin{pmatrix} -2 \\ 3 \end{pmatrix} : y = -\frac{3}{2}x$$

$$\mathbb{c}_1 e^t \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \mathbb{c}_2 e^{2t} \begin{pmatrix} -2 \\ 3 \end{pmatrix} \overset{t \to \infty}{\approx} \mathbb{c}_2 e^{2t} \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

$$\mathbb{c}_1 e^t \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \mathbb{c}_2 e^{2t} \begin{pmatrix} -2 \\ 3 \end{pmatrix} \overset{t \to -\infty}{\approx} \mathbb{c}_1 e^t \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$



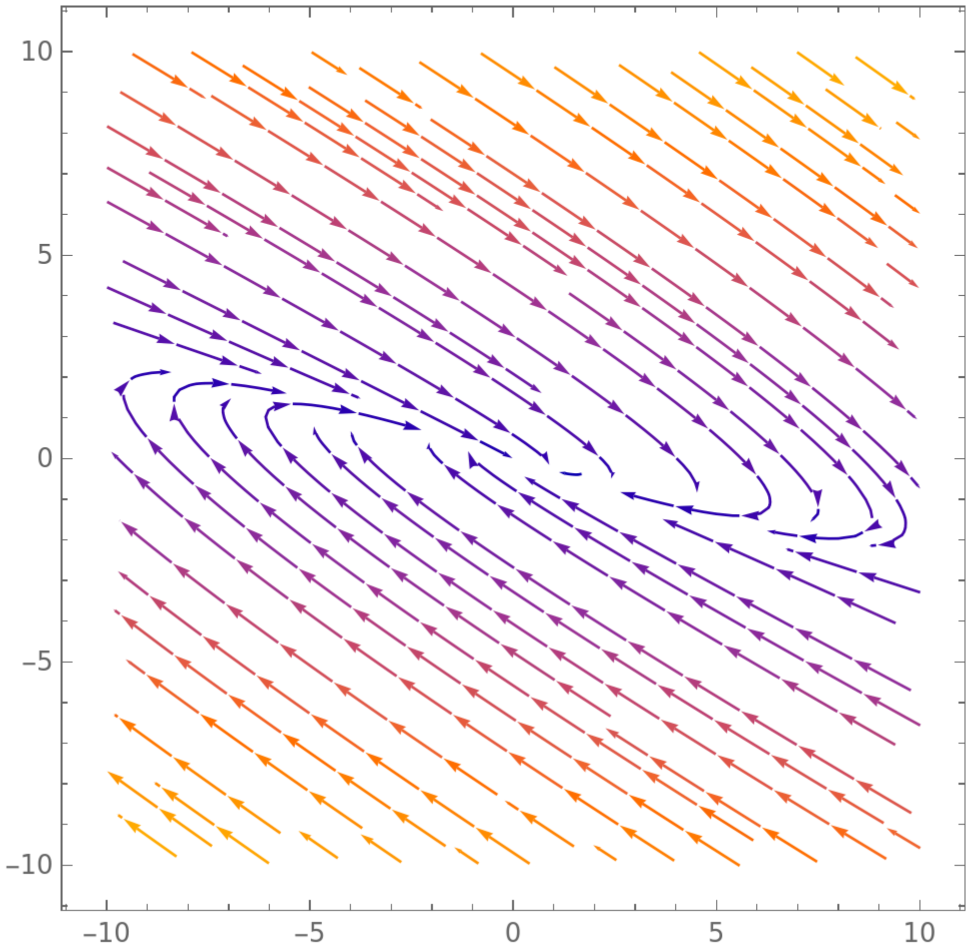
This solution is a nodal source.

Example

$$\vec{x}(t) = \mathbb{c}_1 e^{-t}\begin{pmatrix}-3\\1\end{pmatrix} + \mathbb{c}_2 e^{-2t}\begin{pmatrix}-2\\1\end{pmatrix}$$

$$\mathbb{c}_1 e^{-t}\begin{pmatrix}-3\\1\end{pmatrix} \Rightarrow y = -\frac{1}{3}x$$

$$\mathbb{c}_2 e^{-2t}\begin{pmatrix}-2\\1\end{pmatrix} \Rightarrow y = -\frac{1}{2}x$$



This solution is called a nodal sink.

$$\mathbb{c}_1 e^{-t}\begin{pmatrix}-3\\1\end{pmatrix} + \mathbb{c}_2 e^{-2t}\begin{pmatrix}-2\\1\end{pmatrix} \overset{t\to\infty}{\approx} \mathbb{c}_1 e^{-t}\begin{pmatrix}-3\\1\end{pmatrix} \approx 0$$

$$\mathbb{c}_1 e^{-t}\begin{pmatrix}-3\\1\end{pmatrix} + \mathbb{c}_2 e^{-2t}\begin{pmatrix}-2\\1\end{pmatrix} \overset{t\to-\infty}{\approx} \mathbb{c}_2 e^{-2t}\begin{pmatrix}-2\\1\end{pmatrix}$$

Example

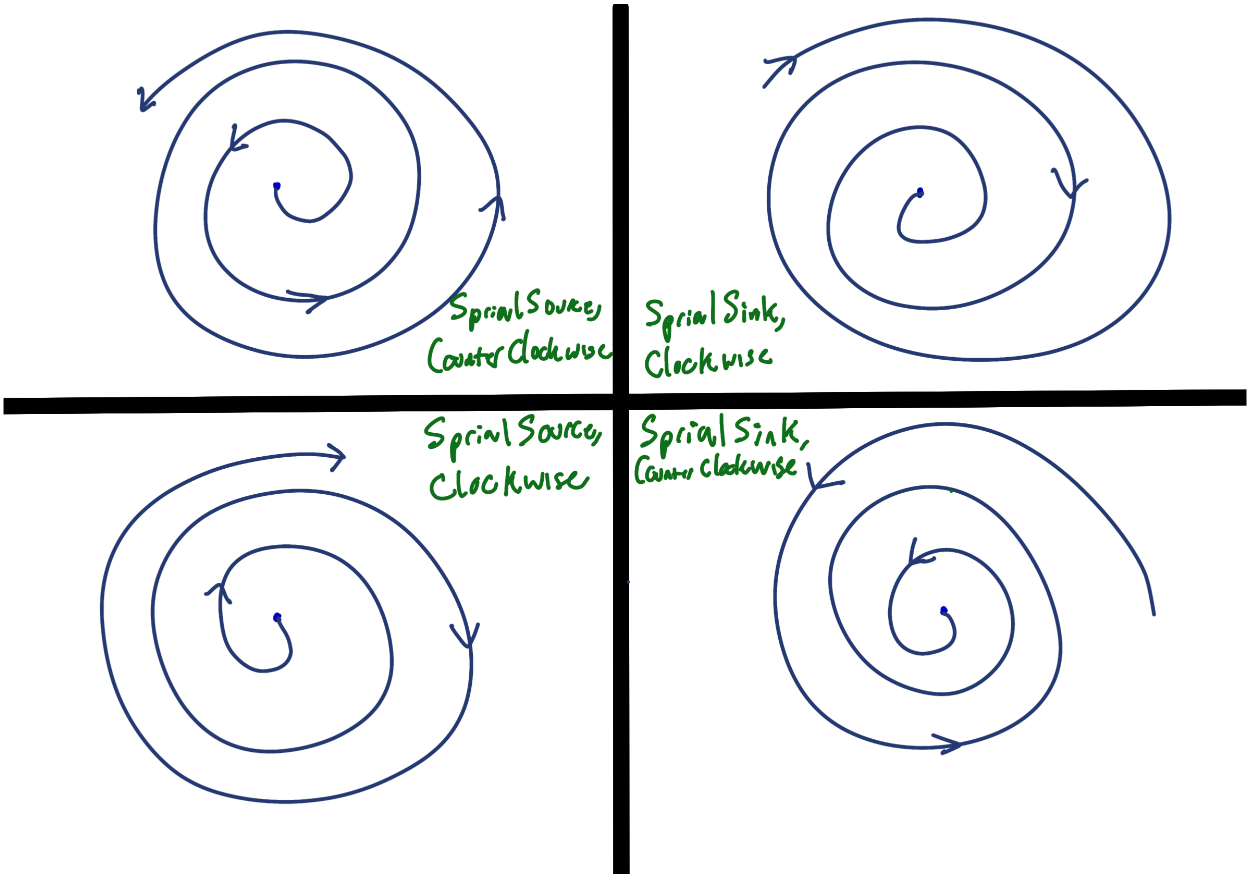
$$\vec{x}(t) = e^{(1+i)t}\begin{pmatrix}-i\\1\end{pmatrix} = e^{t}(\cos(t) + i\sin(t))\begin{pmatrix}-i\\1\end{pmatrix} = e^{t}\begin{pmatrix}(-\cos(t)i + \sin(t))\\(\cos(t) + i\sin(t))\end{pmatrix}$$

Solution

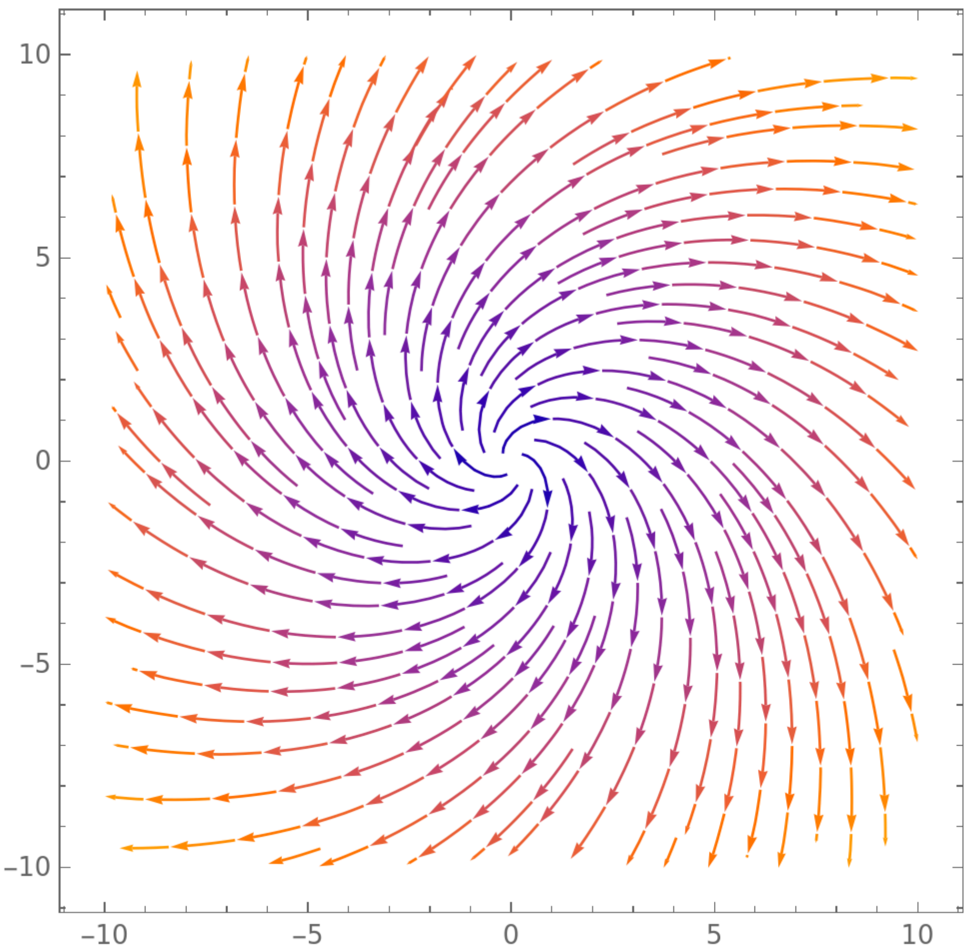
$$\vec{x}(t) = \mathbb{c}_1 e^{t}\begin{pmatrix}-\cos(t)\\\sin(t)\end{pmatrix} + \mathbb{c}_2 e^{t}\begin{pmatrix}\sin(t)\\\cos(t)\end{pmatrix}$$

There is a unique stationary solution at the origin.

This solution is a spiral, so it must be one of these:



It is a source since $e^t \to \infty$ as $t \to \infty$. Since $\frac{\mathrm{d}y}{\mathrm{d}t}\Big|_{(1,0)} = -1$, and $-1 < 0$, it is clockwise.



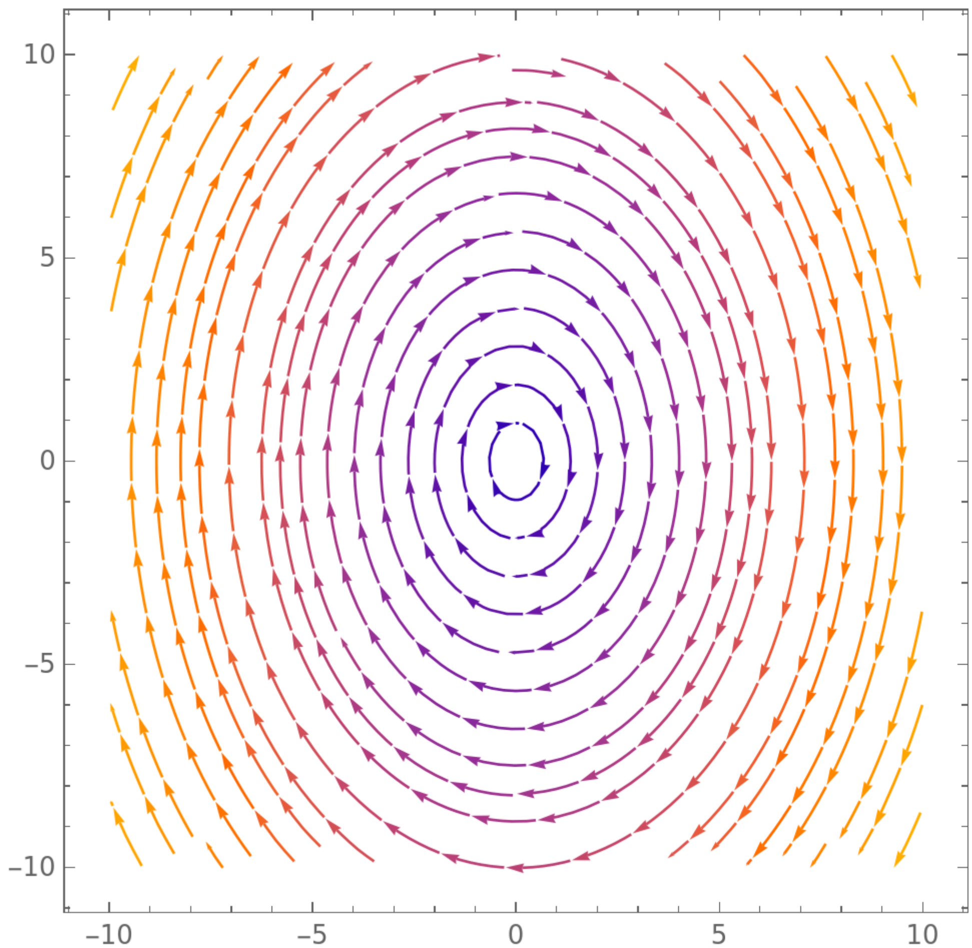
Example

$$\vec{x}(t) = \mathbb{c}_1 e^{2i} \begin{pmatrix} -i \\ 2 \end{pmatrix} + \mathbb{c}_1 e^{-2i} \begin{pmatrix} i \\ 2 \end{pmatrix} = \cdots = \mathbb{c}_1 \begin{pmatrix} \sin(2t) \\ 2\cos(2t) \end{pmatrix} + \mathbb{c}_2 \begin{pmatrix} -\cos(2t) \\ 2\sin(2t) \end{pmatrix}$$

Solution

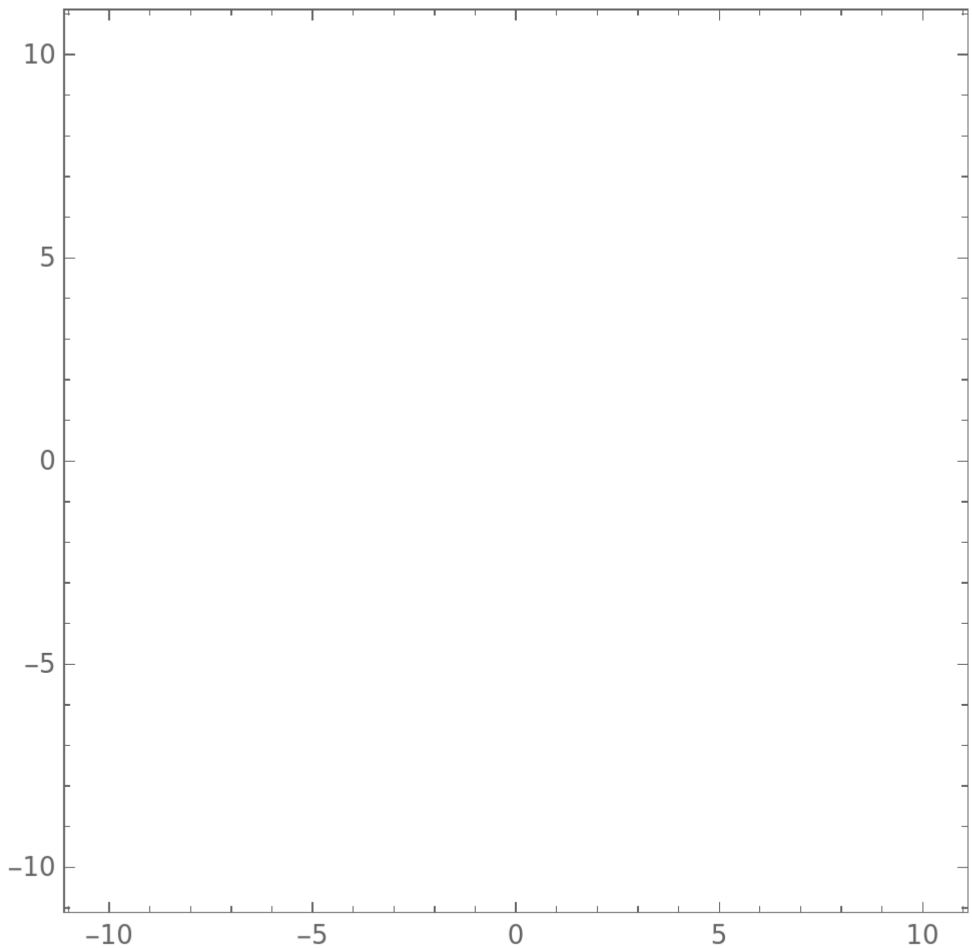
$\frac{\mathrm{d}y}{\mathrm{d}x} = -4x \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}\Big|_{(1,0)} = -4 < 0 \Rightarrow$ clockwise

It is a bunch of ellipses.



Example
$\vec{x}' = \vec{0}$

Solution
The zero solution (just a bunch of infinitely small dots):

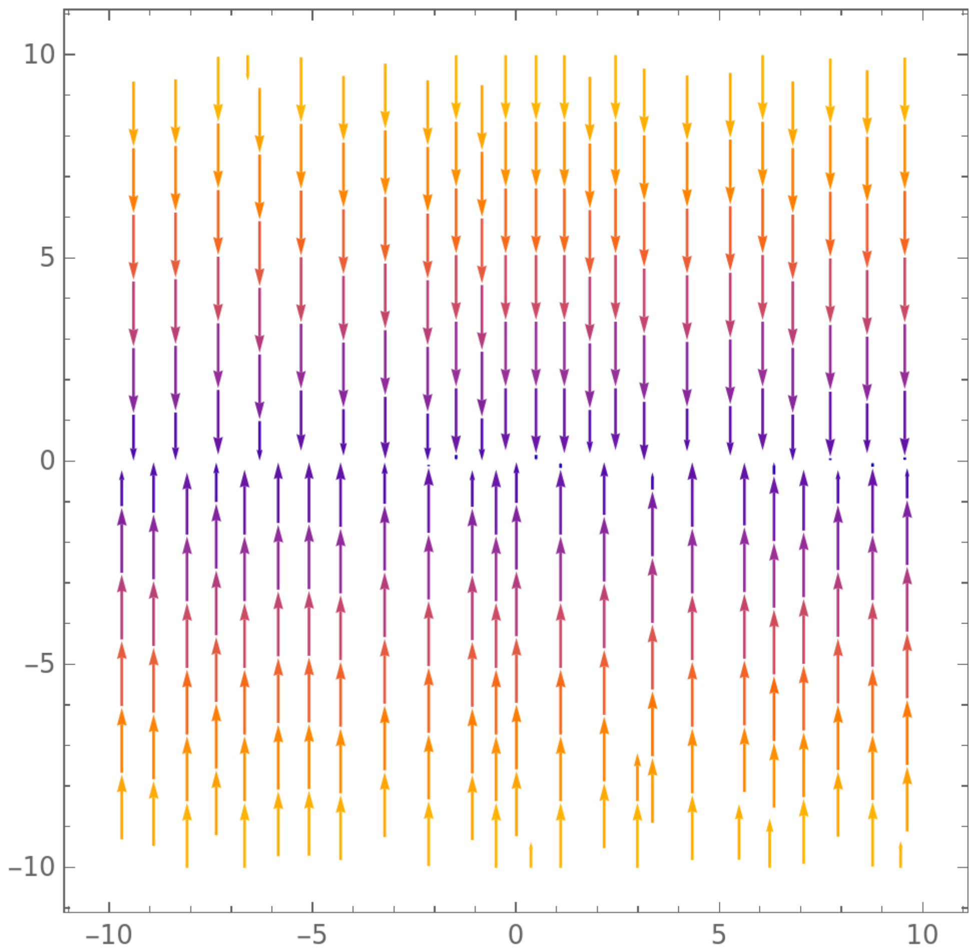


Example

$$\vec{x}(t) = \mathbb{c}_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mathbb{c}_2 e^{-t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Solution

There are stationary solutions for all x when $y = 0$. Then, there are vertical orbits along the y-axis.



Poincare-Bendixon Theorem

Let R be a closed and bounded region of the xy-plane. Suppose $f(x,y)$ and $g(x,y)$ have continuous first partials over an open region containing R. Assume a solution $x(t), y(t)$ to a system of equations:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x,y), \frac{\mathrm{d}y}{\mathrm{d}t} = g(x,y)$$

remains in R for all future t. Suppose further that R contains no stationary curve. Then, either the orbit $(x(t), y(t))$ is itself a closed curve, or it spirals into a simple closed curve, which is itself an orbit of a periodic solution. Therefore, any such system has a periodic solution.

If you are within a region and remain in that region, you will eventually have a periodic or stationary solution.

Example

Prove that the equation has a nontrivial periodic solution:

$$z'' + \left(z^2 + 2z'^2 - 1\right)z' + z = 0$$

Solution:
Form a system:

$$x = z, y = z'$$
$$\Rightarrow \begin{cases} x' = y \\ y' = -(x^2 + 2y^2 - 1)y - x \end{cases}$$

Let

$$f(x, y) = y$$
$$g(x, y) = -(x^2 + 2y^2 - 1)y - x$$

These are C^1 over $\mathbb{R}$.

Find stationary solutions:

$$\begin{cases} 0 = x' = y \\ 0 = y' = -(x^2 + 2y^2 - 1)y - x \end{cases} \Rightarrow \begin{cases} y = 0 \\ 0 = -x \end{cases} \Rightarrow x = y = 0$$

So, when forming a region, we must avoid $(0, 0)$.

Let's assume a disk.

$$\frac{\mathrm{d}}{\mathrm{d}t}(x^2 + y^2) = 2xx' + 2yy' = 2xy + 2y^2(-x^2 - 2y^2 + 1) - 2yx = 2y^2(-x^2 - 2y^2 + 1)$$

If $x^2 + y^2 \geq 1$ then $-x^2 - 2y^2 + 1 < -y^2 \leq 0 \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}(x^2 + y^2) \leq 0$. Therefore, if the disk is of radius 1, pieces will stay within the disk.

If $x^2 + y^2 \leq \frac{1}{2} \Rightarrow 2x^2 + 2y^2 \leq 1 \Rightarrow x^2 \leq 1 - x^2 - 2y^2 \Rightarrow 0 \leq \frac{\mathrm{d}}{\mathrm{d}t}(x^2 + y^2)$, so if the disk is above $\frac{1}{2}$, it cannot enter $\frac{1}{2}$

Therefore take:

$$R = \left\{ (x, y) \in \mathbb{R}^2 \,\middle|\, \frac{1}{3} \leq x^2 + y^2 \leq 2 \right\}$$

R is closed and bounded, and if $(x(0), y(0)) \in R$ then $(x(t), y(t)) \in R$ by the previous statements.

According to the Poincare-Bendixon theorem, a nontrivial periodic solution exists.

Example

$$\begin{cases} x' = x(1 - 4x^2 - y^2) - \frac{1}{2}y(1 + x) \\ y' = y(1 - 4x^2 - y^2) + 2x(1 + x) \end{cases}$$

Show that a nontrivial periodic solution exists.

Solution
Let $R = \left\{ (x, y) \in \mathbb{R}^2 \,\middle|\, \frac{1}{2} \leq 4x^2 + y^2 \leq \frac{3}{2} \right\}$

R is a closed and bounded region.

$$\frac{\mathrm{d}}{\mathrm{d}t}(4x^2 + y^2) = (8x^2 + 2y^2)(1 - (4x^2 + y^2))$$

If (x, y) satisfies $1 \leq 4x^2 + y^2 \leq \frac{3}{2}$ then $\frac{d}{\mathrm{d}t}(4x^2 + y^2) \leq 0$, thus $4x^2 + y^2$ decreases as a function of t.

If (x, y) satisfies $\frac{1}{2} \leq 4x^2 + y^2 \leq 1$ then $\frac{d}{\mathrm{d}t}(4x^2 + y^2) \geq 0$, thus $4x^2 + y^2$ increases as a function of t.

Thus if $(x(0), y(0)) \in R$, then $\forall t, (x(t), y(t)) \in R$

Also, there is no stationary point in R (proven in class), and they have continuous first partials. By the Poincare-Bendixon theorem, a nontrivial periodic solution exists.

Can we find this solution?

We know that it lays on $4x^2 + y^2 = 1$.

$$\begin{cases} x = \frac{1}{2}\cos(f(t)) \\ y = \sin(f(t)) \end{cases} \Rightarrow x' = -\frac{1}{2}\sin(f(t))f'(t) = -\frac{1}{2}\sin(f(t))\left(1 + \frac{1}{2}\cos(f(t))\right) \Rightarrow \left\{ f'(t) = 1 + \tfrac{1}{2}\cos(f(t)) \right.$$

Therefore $f(t) \to 2\arctan\left(\sqrt{3}\tan\left(\frac{1}{4}\left(\sqrt{3}t + 2\sqrt{3}\mathbb{c}_1\right)\right)\right)$. Then, solve for the $\mathbb{c}_1$ by plugging it in again.

Example

$$\begin{cases} x' = -y + x(1 - x^2 - y^2) \\ y' = x + y(1 - x^2 - y^2) \end{cases}$$

Solution

$$\frac{\mathrm{d}}{\mathrm{d}t}(x^2 + y^2) = 2xx' + 2yy' = 2(x^2 + y^2)(1 - x^2 - y^2)$$

There is a single stationary solution at $(0, 0)$ (via multiplication by y and x and then subtraction).

Let $R = \left\{(x, y) \in \mathbb{R} \,\middle|\, \frac{1}{2} \leq x^2 + y^2 \leq \frac{3}{2}\right\}$, a closed and bounded set. $(0, 0) \notin R$. Thus, a nontrivial periodic solution exists.

$$\begin{cases} x = -\cos(t) \\ y = \sin(t) \end{cases} \Rightarrow \text{satisfies the problem}$$

Therefore, it is a solution. Generally, see above and have t instead be $f(t)$