ENEE382 - 0102

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Contents

1. Electrostatics

A charge is a piece of matter that attracts and/or repels other charges.

The SI unit of the charge is the coulomb.

1.1. Charge Properties

1. The sum of all electric charges in a closed system is constant.
   - Charge is conserved
2. Charge is quantized
   - Charge is quantized into units of the elementary charge: $e = 1.6 \times 10^{-19}\ \mathrm{C}$

1.2. Charge density

Line charge

$$\rho_L = \lim_{\Delta l \to 0} \frac{\Delta q}{\Delta \ell} /\!/\!/ \frac{\mathrm{C}}{\mathrm{m}}$$

Surface charge

$$\rho_S = \lim_{\Delta a \to 0} \frac{\Delta q}{\Delta a} /\!/\!/ \frac{\mathrm{C}}{\mathrm{m}^2}$$

Volume charge

$$\rho_V = \lim_{\Delta v \to 0} \frac{\Delta q}{\Delta v} /\!/\!/ \frac{\mathrm{C}}{\mathrm{m}^3}$$

1.3. Current

Current occurs when charges move.

$$I = \frac{\mathrm{d}Q}{\mathrm{d}t}$$

1.4. Coulomb's Law

$$\vec{F}_{1 \leftarrow 2} = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{R^2} \hat{R}$$

$\hat{R}$ is a unit vector pointing from charge 2 from charge 1.

1.5. SI Units

Quantity	Unit
Length	m
Mass	Kg
Time	S
Current	A

$$\varepsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F/m}$$

1.6. Electric Field

$$\begin{aligned} \vec{F} &= \sum_{k=1}^{n} \frac{1}{4\pi\varepsilon_0} \frac{q_k q_t}{R_k^2} \widehat{R_k} \\ &= q_t \sum_{k=1}^{n} \frac{1}{4\pi\varepsilon_0} \frac{q_k}{R_k^2} \widehat{R_k} \end{aligned}$$

$$= q_t \vec{E}$$

Therefore, $\vec{F} = q_t \vec{E}$, where $\vec{E}$ is the electric field.

Now, to operate over a continuous charge *distribution* ρ_V, over a small region:

$$q \approx \rho_V(x, y, z) \underbrace{(\Delta x \times \Delta y \times \Delta z)}_{\Delta V}$$

Over a single element, then, the force on $\mathcal{E}_t$ is:

$$\Delta \vec{F} = \frac{1}{4\pi\varepsilon_0} \frac{q_t(\rho_V(x, y, z))\Delta V}{R^2} \hat{R}$$

Where $\vec{R} = \vec{r} - \vec{r}'$, where $\vec{r}$ is the position of $\mathcal{E}_t$ and $\vec{r}' = (x, y, z)$

Then:

$$\vec{F} = \sum \frac{1}{4\pi\varepsilon_0} \frac{q_t q}{R^2} \hat{R}$$
$$= q_t \frac{1}{4\pi\varepsilon_0} \sum \frac{q_t q}{R^2} \hat{R}$$

Then,

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho_V(\vec{r}')}{R(r')^2} \hat{R}(r') \,\mathrm{d}v'$$

Similarly, over a surface:

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \int_S \frac{\rho_S(\vec{r}')}{R(r')^2} \hat{R}(r') \,\mathrm{d}s'$$

And over a line:

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \int_L \frac{\rho_L(\vec{r}')}{R(r')^2} \hat{R}(r') \,\mathrm{d}\ell'$$

1.6.1. Example

ρ_ℓ is constant.

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \int_\ell \frac{\rho_\ell(r')}{R^2} \hat{R} \,\mathrm{d}\ell'$$
$$= \frac{\rho_\ell}{4\pi\varepsilon_0} \int_\ell \frac{1}{R^2} \hat{R} \,\mathrm{d}\ell'$$

Now, see that $\vec{r}' = (x, 0, 0)$, and $\vec{r} = (0, 0, d)$, alternatively written $\vec{r}' = x'\hat{x}$ and $\vec{r} = d\hat{z}$.

Therefore, $R = \sqrt{d^2 + x'^2}$, and

$$\hat{R} = \frac{1}{\sqrt{d^2 + x'^2}} (-x, 0, d)$$

Then, integrate from $-L$ to L over x'

$$= \frac{\rho_\ell}{4\pi\varepsilon_0} \int_{-L}^{L} \frac{(-x, 0, d)}{(d^2 + x'^2)^{3/2}} \,\mathrm{d}x$$

Notice that x is an odd function, and is then multiplied by an even function of x', and therefore the integral does not need that component, as you are integrating symmetrically across an odd function

$$= \frac{\rho_\ell}{4\pi\varepsilon_0} \int_{-L}^{L} \frac{(0,0,d)}{\left(d^2 + x'^2\right)^{3/2}} \,\mathrm{d}x$$

$$\overset{\text{even function, linearity}}{=} \frac{2\rho_\ell}{4\pi\varepsilon_0}(0,0,d) \int_{0}^{L} \frac{1}{\left(d^2 + x'^2\right)^{3/2}} \,\mathrm{d}x$$

$$= \frac{l\rho_l}{2\pi\varepsilon_0 d\sqrt{d^2 + l^2}}$$

1.7. Force

The force on a charge at position $\vec{r}$ with charge q is $\vec{F}(\vec{r}) = q\vec{E}(\vec{r})$.

1.8. Divergence

$$\nabla \cdot \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

1.8.1. Example

Divergence of the electric field for a point charge.

Let use express this in spherical coordinates:

$$\vec{E}(r,\theta,\varphi) = \frac{q}{4\pi\varepsilon_0 r^2}\hat{r}$$

$$\begin{aligned}
\nabla \cdot \vec{E} &= \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 E_r\right] \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{q}{4\pi\varepsilon_0 r^2}\right] \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left[\frac{q}{4\pi\varepsilon_0}\right] \\
&\overset{\text{Assume } r \neq 0}{=} \frac{1}{r^2} \cdot 0 \\
&= 0
\end{aligned}$$

This is true while $r \neq 0$.

Let V be a spherical volume of radius R.

$$\int_V \nabla \cdot \vec{E} \,\mathrm{d}v$$

Let S be the surface of a sphere of radius R. By the divergence theorem:

$$= \oint_S \vec{E} \cdot \mathrm{d}\vec{a}$$

Note that

$$\mathrm{d}\vec{a} = R^2 \sin(\theta) \,\mathrm{d}\theta \,\mathrm{d}\varphi \hat{r}$$

Then:

$$
\begin{aligned}
&= \oint_S \vec{E} \cdot \hat{r} R^2 \sin(\theta) \,\mathrm{d}\theta \,\mathrm{d}\varphi \\
&= \oint_S \frac{q}{4\pi\varepsilon_0 R^2} R^2 \sin(\theta) \,\mathrm{d}\theta \,\mathrm{d}\varphi \\
&= \frac{q}{4\pi\varepsilon_0} \oint_S \sin(\theta) \,\mathrm{d}\theta \,\mathrm{d}\varphi \\
&= \frac{q}{4\pi\varepsilon_0} \int_0^{2\pi} \int_0^{\pi} \sin(\theta) \,\mathrm{d}\theta \,\mathrm{d}\varphi \\
&= \frac{q}{4\pi\varepsilon_0} \int_0^{2\pi} 2 \,\mathrm{d}\varphi \\
&= \frac{q}{4\pi\varepsilon_0} 4\pi \\
&= \frac{q}{\varepsilon_0}
\end{aligned}
$$

So we know the integral, and the point at everywhere but $r = 0$, and therefore:

$$\nabla \cdot \vec{E} = \frac{q}{\varepsilon_0} \delta(\vec{r}) = \frac{q}{\varepsilon_0} \delta(x)\delta(y)\delta(z)$$

δ is a object such that

$$\int_{-\infty}^{\infty} \delta(x) \,\mathrm{d}x = 1$$

and $\forall x \neq 0, \delta(x) = 0$.

Then,

$$\int_{-\infty}^{\infty} f(x)\delta(x) \,\mathrm{d}x = f(0)$$

1.9. Inverse square law is unique for a point charge

Note that if you have a function defined $\vec{A}(\vec{r}) = \frac{1}{r^2}\hat{r}$, then $\nabla \cdot \vec{A}(\vec{r}) = 4\pi\delta(\vec{r})$.

Let $\vec{A}(\vec{r}) = f(|\vec{r}|)\hat{r}$ ($\vec{A}$ is radially symmetric). If $\nabla \cdot \vec{A}(\vec{r}) = c\delta(\vec{r})$, then $\vec{A}(\vec{r}) = \frac{c}{4\pi r^2}\hat{r}$.

1.10. Gauss's law

$$
\begin{aligned}
\nabla \cdot \vec{E} &= \frac{1}{4\pi\varepsilon_0} \nabla \cdot \int_V \frac{\rho(\vec{r}')}{R^2} \hat{R} \,\mathrm{d}v' \\
&= \frac{1}{4\pi\varepsilon_0} \int_V \nabla \cdot \frac{\rho(\vec{r}')}{R^2} \hat{R} \,\mathrm{d}v' \\
&= \frac{1}{4\pi\varepsilon_0} \int_V \rho(\vec{r}') \left(\nabla \cdot \frac{1}{R^2} \hat{R} \right) \mathrm{d}v'
\end{aligned}
$$

This works because ∇ is differentiating with respect to (x, y, z), but we are integrating over (x', y', z').

$$
\begin{aligned}
&= \frac{1}{4\pi\varepsilon_0} \int_V \rho(\vec{r}')(4\pi\delta(\vec{r} - \vec{r}')) \,\mathrm{d}v' \\
&= \frac{1}{\varepsilon_0} \int_V \rho(\vec{r}')(\delta(\vec{r} - \vec{r}')) \,\mathrm{d}v'
\end{aligned}
$$

$$= \frac{1}{\varepsilon_0}\rho(\vec{r})$$

Therefore:

$$\begin{aligned}
\nabla \cdot \vec{E} &= \frac{1}{\varepsilon_0}\rho(\vec{r}) \\
\oiiint \nabla \cdot \vec{E} \,\mathrm{d}v' &= \oiiint \frac{1}{\varepsilon_0}\rho(\vec{r}) \,\mathrm{d}v' \\
\oiint \vec{E} \cdot \hat{n} \,\mathrm{d}S &= \frac{1}{\varepsilon_0} \oiiint \rho(\vec{r}) \,\mathrm{d}v' \\
\oiint \vec{E} \cdot \hat{n} \,\mathrm{d}S &= \frac{Q_{\text{enclosed}}}{\varepsilon_0}
\end{aligned}$$

1.11. Curl of the electric field

Let

$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}\hat{r}$$

Then

$$\int_{\vec{a}}^{\vec{b}} \vec{E} \cdot \mathrm{d}\vec{\ell}$$

See that

$$\mathrm{d}\vec{\ell} = \mathrm{d}r\hat{r} + r\,\mathrm{d}\theta\hat{\theta} + r\sin(\theta)\,\mathrm{d}\varphi\hat{\varphi}$$

Therefore:

$$\begin{aligned}
&= \int_{\vec{a}}^{\vec{b}} \frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}\,\mathrm{d}r \\
&= \frac{q}{4\pi\varepsilon_0}\int_a^b \frac{1}{r^2}\,\mathrm{d}r \\
&= \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)
\end{aligned}$$

Notice that the integral does not depend on the path one takes through the electric field. Therefore, the integral from a point to itself is zero.

Therefore, for a closed path P:

$$\oint_P \vec{E} \cdot \mathrm{d}\vec{\ell} = 0$$

This is true for all electric fields as all electric fields are composed from individual point charges. This implies that the electric field is a conservative field.

Then, take the surface integral over a surface S where the boundary of the surface is the closed path P.

$$\oiint_S \nabla \times \vec{E} \,\mathrm{d}\vec{a} = 0$$

Take for example:

$$\oiint_S f(\vec{x})\,\mathrm{d}\hat{x}$$

Since S can be any surface, this implies that $f(\vec{x}) = 0$, for any $\vec{x}$.

By the same principle, $\nabla \times \vec{E} = 0$.

1.12. Planar Symmetry

If $\rho(x, y, z) = \rho(z)$ (or if you can transform the distribution to be in such a form), then you have planar symmetry.

An example of planar symmetry is on an infinite plane with surface charge density σ.

$$\iint \vec{E} \cdot \mathrm{d}\vec{a} = \frac{Q_{\text{enclosed}}}{\varepsilon_0}$$

Under planar symmetry:

$$\vec{E}(x, y, z) = \vec{E}(z) = E(z)\hat{z}$$

Intuitively, this is because you can always do a change of variables, since $\rho(x, y, z) = \rho(z)$, and then, since it is constant on some particular plane, there are always opposing forces.

Consider a generalized cylinder extending from $-z$ to z.

$$\int \vec{E} \cdot \mathrm{d}\vec{\ell} = \int \text{top} + \int \text{bottom} + \int \text{sides}$$

Notice that $\vec{E} \cdot \mathrm{d}\vec{a} = 0$ for the sides of a cylinder. This is because $\hat{E}(z) = \pm\hat{z}$ but $\widehat{\mathrm{d}\vec{a}}$ is not at all in the z direction. Therefore:

$$\int \text{sides} = 0$$

Further:

$$\int \text{top} = \int_{\text{top}} E(z)\,\mathrm{d}x\,\mathrm{d}y = E(z) \int_{\text{top}} \mathrm{d}x\,\mathrm{d}y = E(z)A$$

$$\int \text{bottom} = \int_{\text{bottom}} E(-z)(-1)\,\mathrm{d}x\,\mathrm{d}y = -E(-z) \int_{\text{bottom}} \mathrm{d}x\,\mathrm{d}y = -E(-z)A$$

$$\int \text{top} + \int \text{bottom} = A(E(z) - E(-z))$$

Furthermore, if you assume you have inversion symmetry ($\rho(z) = -\rho(-z)$, and thereby $E(z) = -E(-z)$):

$$\int \vec{E} \cdot \mathrm{d}\vec{\ell} = 2AE(z)$$

Plugging back into Gauss's law:

$$\oint \vec{E} \cdot \mathrm{d}\vec{a} = \frac{Q_{\text{enclosed}}}{\varepsilon_0}$$

Let us use the example of the planar symmetric on an infinite plane with surface charge density σ. Then:

$$2AE(z) = \frac{\sigma A}{\varepsilon_0}$$

$$E(z) = \frac{\sigma}{2\varepsilon_0}$$

1.13. Spherical Symmetry Example

$$\oint \vec{E} \cdot \mathrm{d}\vec{a} = \frac{Q_{\text{enclosed}}}{\varepsilon_0}$$

Assume spherical symmetry:

$$\rho(r, \theta, \varphi) = \rho(r)$$

Therefore

$$\vec{E}(r, \theta, \varphi) = \vec{E}(r) = E(r)\hat{r}$$

You can see $\vec{E}(r) = E(r)\hat{r}$ by considering a series of circles, all of which sum to zero in the direction that is not $\hat{r}$.

Consider the surface that is a sphere of radius r.

$$\begin{aligned}
&\int_0^{2\pi} \int_0^{\pi} E(r) r^2 \sin(\theta) \,\mathrm{d}\theta \,\mathrm{d}\varphi \hat{r} \\
&= E(r) r^2 \hat{r} \int_0^{2\pi} \int_0^{\pi} \sin(\theta) \,\mathrm{d}\theta \,\mathrm{d}\varphi \\
&= E(r) r^2 \hat{r} 4\pi \\
&= 4\pi r^2 E(r) \hat{r}
\end{aligned}$$

Applying Gauss's law then implies:

$$\begin{aligned}
4\pi r^2 E(r) &= \frac{Q_{\text{enclosed}}}{\varepsilon_0} \\
E(r) &= \frac{Q_{\text{enclosed}}}{4\pi\varepsilon_0 r^2}
\end{aligned}$$

Assume you have a spherical shell with total charge Q and radius R.

When $r > R$, $Q_{\text{enclosed}} = Q$ implies

$$E(r) = \frac{Q}{4\pi\varepsilon_0 r^2}$$

When $r < R$, $Q_{\text{enclosed}} = 0$ implies

$$E(r) = 0$$

1.14. Showing Kirchhoff's Voltage Law

We know that

$$\nabla \times \vec{E} = 0$$

$$\vec{E} = -\nabla V(\vec{r})$$

$$W = -\int_{\vec{a}}^{\vec{b}} \vec{F} \cdot \mathrm{d}\vec{\ell}$$

$$
\begin{aligned}
&= -q\int_{\vec{a}}^{\vec{b}} \vec{E}\cdot \mathrm{d}\vec{\ell} \\
&= q\int_{\vec{a}}^{\vec{b}} \nabla V\cdot \mathrm{d}\vec{\ell} \\
&= q\int_{\vec{a}}^{\vec{b}} \nabla V\cdot \mathrm{d}\vec{\ell} \\
&= q\left(V\left(\vec{b}\right) - V(\vec{a})\right)
\end{aligned}
$$

Therefore

$$\frac{W}{q} = V\left(\vec{b}\right) - V(\vec{a})$$

Furthermore, let P be some closed path.

$$V = -\oint_P \vec{E}\cdot \mathrm{d}\vec{\ell} = 0$$

Split this path into parts P_i:

$$0 = \sum_i \int_{P_i} \vec{E}\cdot \mathrm{d}\vec{\ell} = \sum_i \Delta V_i$$

We have now shown KCL.

1.15. Point charge electric potential

Assume you have a point charge of charge q is at the origin.

Then, let $\vec{r}$ be the place where we which to see the potential, and we measure the potential relative to a point at infinity, since the electric field decays to zero at infinity:

$$
\begin{aligned}
V &= -\int_{\vec{\infty}}^{\vec{r}} \vec{E}(\vec{r}\,')\cdot \mathrm{d}\vec{\ell} \\
&= -\int_{\infty\hat{r}}^{\vec{r}} \vec{E}(\vec{r}\,')\cdot \mathrm{d}\vec{\ell} \\
&= -\int_{\infty}^{r} \frac{q}{4\pi\varepsilon_0 r'^2}\hat{r}'\cdot\hat{r}'\,\mathrm{d}r \\
&= -\frac{q}{4\pi\varepsilon_0}\int_{\infty}^{r} \frac{1}{r'^2}\,\mathrm{d}r \\
&= -\frac{q}{4\pi\varepsilon_0}\left[-\frac{1}{r'}\right]_{\infty}^{r} \\
&= -\frac{q}{4\pi\varepsilon_0}\left(-\frac{1}{r} - \left(-\frac{1}{\infty}\right)\right) \\
&= \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r} + \left(-\frac{1}{\infty}\right)\right) \\
&= \frac{q}{4\pi\varepsilon_0 r}
\end{aligned}
$$

We can shift this to any point $\vec{r}\,'$, so $V(\vec{r}) = \frac{q}{4\pi\varepsilon_0|\vec{r}-\vec{r}\,'|}$

If we have multiple point charges, we can just sum:

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \sum_i \frac{q_i}{|\vec{r} - \vec{r}_i|}$$

1.16. Distribution integral for electric field

Consider a volume V with a continuous charge distribution $\rho_v(\vec{r})$.

Then:

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \oiiint_V \frac{\rho_v(\vec{r}')}{|\vec{r} - \vec{r}'|} \,\mathrm{d}v'$$

1.17. Flat circular disk with uniform surface charge

Consider a flat circular disk of radius b with a uniform surface charge λ. Call the disk S. Calculate the potential at a point height z over the center of the disk.

$$\begin{aligned} \vec{r} &= z\hat{z} \\ \vec{r}' &= s'\hat{s} = (s'\cos(\varphi'), s'\sin(\varphi'), 0) \end{aligned}$$

$$\begin{aligned} V(\vec{r}) &= \frac{1}{4\pi\varepsilon_0} \oiint_S \rho_S \frac{\vec{r}'}{|\vec{r} - \vec{r}'|} \,\mathrm{d}\vec{a} \\ &= \frac{1}{4\pi\varepsilon_0} \int_0^{2\pi} \int_0^b \frac{\lambda}{|z\hat{z} - s'\hat{z}|} s' \,\mathrm{d}s' \,\mathrm{d}\varphi' \\ &= \frac{\lambda}{4\pi\varepsilon_0} \int_0^{2\pi} \int_0^b \frac{s'}{\sqrt{z^2 + s'^2}} \,\mathrm{d}s' \,\mathrm{d}\varphi' \\ &= \frac{\lambda}{2\varepsilon_0} \left(\sqrt{z^2 + b^2} - |z| \right) \end{aligned}$$

To find the electric field:

$$\begin{aligned} \vec{E} &= -\nabla V = -\hat{z}\frac{\mathrm{d}V}{\mathrm{d}z} \\ &= -\hat{z}\frac{\lambda}{2\varepsilon_0}\left(\frac{z}{\sqrt{z^2 + b^2}} - \operatorname{sgn}(z) \right) \\ &= -\hat{z}\frac{\lambda}{2\varepsilon_0}\left(\frac{\operatorname{sgn}(z)|z|}{\sqrt{z^2 + b^2}} - \operatorname{sgn}(z) \right) \\ &= -\operatorname{sgn}(z)\hat{z}\frac{\lambda}{2\varepsilon_0}\left(\frac{|z|}{\sqrt{z^2 + b^2}} - 1 \right) \end{aligned}$$

Notice that this field is discontinuous at the surface charge distribution (the vector flips sign). This is true in general!

1.18. Surface charge distribution

Consider a charged surface S, and some point p on that surface. Construct a generalized cylinder from a surface of an area A of length ℓ centered at the point p, small enough so that the surface charge density is a constant, ρ_S. Consider the electric field $\vec{E}^{\text{top}}$ on the top of the cylinder, and $\vec{E}^{\text{bottom}}$ at the bottom of the cylinder.

Then, by Gauss's law:

$$\oiint \vec{E} \cdot \mathrm{d}\vec{a} = \frac{Q_{\text{enclosed}}}{\varepsilon_0}$$

$$\oiint_{\text{top}} \vec{E} \cdot \mathrm{d}\vec{a} + \oiint_{\text{bottom}} \vec{E} \cdot \mathrm{d}\vec{a} + \oiint_{\text{sides}} \vec{E} \cdot \mathrm{d}\vec{a} = \frac{Q_{\text{enclosed}}}{\varepsilon_0}$$

We will be doing this at the limit for a very small cylinder, so $\ell \to 0$. Therefore, $\oiint_{\text{sides}} \vec{E} \cdot \mathrm{d}\vec{a} \to 0$.

$$\oiint_{\text{top}} \vec{E} \cdot \mathrm{d}\vec{a} + \oiint_{\text{bottom}} \vec{E} \cdot \mathrm{d}\vec{a} = \frac{Q_{\text{enclosed}}}{\varepsilon_0}$$

Split the electric filed into two vectors, $\vec{E} = \vec{E}_{\perp} + \vec{E}_{\parallel}$, so that $\vec{E}_{\perp} \cdot \hat{n} = E_{\text{perp}}$ (the perpendicular vector is in line with the normal vector).

Therefore:

$$\vec{E}_{\perp}^{\text{top}} \oiint_{\text{top}} \mathrm{d}a - \vec{E}_{\perp}^{\text{bottom}} \oiint_{\text{bottom}} \mathrm{d}a = \frac{Q_{\text{enclosed}}}{\varepsilon_0}$$

$$\vec{E}_{\perp}^{\text{top}} A - \vec{E}_{\perp}^{\text{bottom}} A = \rho_S \frac{A}{\varepsilon_0}$$

$$\vec{E}_{\perp}^{\text{top}} - \vec{E}_{\perp}^{\text{bottom}} = \frac{\rho_S}{\varepsilon_0}$$

So there will always be a discontinuity in the electric field after passing through a surface.

Conversely, consider a surface charge distribution, and then consider a centered perpendicular rectangle through the surface charge distribution.

Then:

$$\oint \vec{E} \cdot \mathrm{d}\vec{\ell} = 0$$

Consider the same electric field. Then, on the segment at the top, it's going some direction, and the segment at the bottom is going in the opposite direction. Denote the vector of the top segment to be $\vec{\ell}$. Alternatively, consider the vertical component to have length $2d$ and vector direction $\hat{n}$. Notice that $\vec{E}_{\perp} \cdot \hat{n} = \left|\vec{E}_{\perp}\right|$. Then, take the integral over the individual segments. Assume all vectors here have length small enough so that the change in the electric field over the length is irrelevant. Then, take the integral:

$$\vec{\ell} \cdot \vec{E}_{\parallel}^{\text{top}} - d\hat{n} \cdot \vec{E}_{\perp}^{\text{top}} - d\hat{n} \cdot \vec{E}_{\perp}^{\text{bottom}} - \vec{\ell} \cdot \vec{E}_{\parallel}^{\text{top}} + d\hat{n} \cdot \vec{E}_{\perp}^{\text{top}} + d\hat{n} \cdot \vec{E}_{\perp}^{\text{bottom}} = 0$$

Consider this as $d \to 0$:

$$\vec{\ell} \cdot \vec{E}_{\parallel}^{\text{top}} - \vec{\ell} \cdot \vec{E}_{\parallel}^{\text{top}} = 0$$

$$\vec{\ell} \cdot \vec{E}_{\parallel}^{\text{top}} = \vec{\ell} \cdot \vec{E}_{\parallel}^{\text{top}}$$

Since this was true for any vector $\vec{\ell}$ we have:

$$\vec{E}_{\parallel}^{\text{top}} = \vec{E}_{\parallel}^{\text{bottom}}$$

1.19. Work to put charges in a place

Consider you have two charges q_1 and q_2. First, place them infinitely far away.

Move q_1 to r_1. No force is applied so zero work occurred.

Then, move q_2 to r_2:

$$V = \frac{q_1}{4\pi\varepsilon_0 |r_2 - r_1|}$$

$$W = \frac{q_1 q_2}{4\pi\varepsilon_0 |r_2 - r_1|}$$

If you had 3 charges, then:

$$W = \frac{q_1 q_2}{4\pi\varepsilon_0 |r_2 - r_1|} + \frac{q_1 q_3}{4\pi\varepsilon_0 |r_3 - r_1|} + \frac{q_2 q_3}{4\pi\varepsilon_0 |r_3 - r_2|}$$

Define

$$W_{ij} = \frac{q_i q_j}{4\pi\varepsilon_0 |r_j - r_i|}$$

Notice $W_{ij} = W_{ji}$. Therefore, where W_i is the work done to add the ith charge:

$$W_i = \sum_{j=1}^{i=1} W_{ij}$$

In sum:

$$W = \sum_{i=2}^{n} W_i = \sum_{i=2}^{i-1} \sum_{j=1}^{i-1} W_{ij}$$

Further:

$$W = \sum_{j=1}^{n-1} \sum_{i=j+1}^{n} W_{ji} = \sum_{j=1}^{n-1} \sum_{i=j+1}^{n} W_{ij}$$

Then, we sum the sums:

$$2W = \sum_{i=2}^{i-1} \sum_{j=1}^{i-1} W_{ij} + \sum_{j=1}^{n-1} \sum_{i=j+1}^{n} W_{ij} = \sum_{i \neq j} W_{ij}$$

Therefore,

$$\begin{aligned} W &= \frac{1}{2} \sum_{i \neq j} W_{ij} \\ &= \frac{1}{2} \sum_{i=1}^{n} q_i V_i(\vec{r}_i) \end{aligned}$$

Where $V_i(\vec{r}_i)$ is the potential of all charges except i.

Let us instead take this over a continuous charge distribution:

$$W = \frac{1}{2} \int \rho(\vec{r}) V(\vec{r}) \, \mathrm{d}v$$

We also know that

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} dr'$$

And therefore

$$W = \frac{1}{2} \frac{1}{4\pi\varepsilon_0} \iint \frac{\rho(\vec{r})\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} \, \mathrm{d}v dr'$$

1.19.1. Example

Find the energy required to assemble a uniform spherical surface of radius b and total charge Q which is uniformly distributed.

We know that $\rho(\vec{r}) = \frac{Q}{A} = \frac{Q}{4\pi b^2}$.

Then the potential on the surface is $V = \frac{1}{4\pi\varepsilon_0}\frac{Q}{R} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{b}$.

Then

$$\begin{aligned} W &= \frac{1}{2}\int \rho(\vec{r})V(\vec{r}) \cdot \mathrm{d}\vec{a} \\ &= \frac{1}{2}\int \rho(\vec{r})\frac{1}{4\pi\varepsilon_0}\frac{Q}{b} \cdot \mathrm{d}\vec{a} \\ &= \frac{1}{2}\frac{1}{4\pi\varepsilon_0}\frac{Q}{b}\int \rho(\vec{r}) \cdot \mathrm{d}\vec{a} \end{aligned}$$

And then due to spherical symmetry:

$$\begin{aligned} &= \frac{1}{2}\frac{1}{4\pi\varepsilon_0}\frac{Q}{b}Q \\ &= \frac{Q^2}{8\pi\varepsilon_0 b} \end{aligned}$$

Notice that as $b \to 0$, $W \to \infty$.

1.20. Energy stored in the electric field

Consider the integral over the entire space, when $\rho(\vec{r})$ is only defined for finite $\vec{r}$:

$$\frac{1}{2}\int \rho(\vec{r})V(\vec{r})\,\mathrm{d}\vec{r}$$

Then:

$$\rho(\vec{r}) = \varepsilon_0 \nabla \cdot \vec{E}(\vec{r})$$

$$\begin{aligned} W &= \frac{1}{2}\int \varepsilon_0 \nabla \cdot \vec{E}(\vec{r})V(\vec{r})\,\mathrm{d}r \\ &= \frac{\varepsilon_0}{2}\int \left(\nabla \cdot \vec{E}(\vec{r})\right)V(\vec{r})\,\mathrm{d}r \end{aligned}$$

Then it is true that:

$$\nabla \cdot \left(V(\vec{r})\vec{E}(\vec{r})\right) = V(\vec{r})\left(\nabla \cdot \vec{E}(\vec{r})\right) + \vec{E}(\vec{r}) \cdot (\nabla V(\vec{r}))$$

For any scalar field $V(\vec{r})$ and vector field $\vec{E}(\vec{r})$.

Therefore:

$$V(\vec{r})\left(\nabla \cdot \vec{E}(\vec{r})\right) = \nabla \cdot \left(V(\vec{r})\vec{E}(\vec{r})\right) - \vec{E}(\vec{r}) \cdot (\nabla V(\vec{r}))$$

And further:

$$W = \frac{\varepsilon_0}{2}\int \nabla \cdot \left(V(\vec{r})\vec{E}(\vec{r})\right) - \vec{E}(\vec{r}) \cdot (\nabla V(\vec{r}))\,\mathrm{d}r$$

$$\begin{aligned} &= \frac{\varepsilon_0}{2}\left[\int \nabla \cdot \left(V(\vec{r})\vec{E}(\vec{r})\right) \mathrm{d}r - \int \vec{E}(\vec{r}) \cdot (\nabla V(\vec{r})) \,\mathrm{d}r\right] \\ &= \frac{\varepsilon_0}{2}\left[\oint V(\vec{r})\vec{E}(\vec{r}) \,\mathrm{d}\vec{a} - \int \vec{E}(\vec{r}) \cdot (\nabla V(\vec{r})) \,\mathrm{d}r\right] \end{aligned}$$

Then:

$$\oint V(\vec{r})\vec{E}(\vec{r}) \,\mathrm{d}\vec{a} = 0$$

Intuitively, this is because area scales with r^2, but the electric field and the potential scale with $\frac{1}{r^2}$ and $\frac{1}{r}$ respectively, and $r^2 < r^3$.

$$\begin{aligned} W &= -\frac{\varepsilon_0}{2} \int \vec{E}(\vec{r}) \cdot (\nabla V(\vec{r})) \,\mathrm{d}r \\ &= -\frac{\varepsilon_0}{2} \int \vec{E}(\vec{r}) \cdot \left(-\vec{E}(\vec{r})\right) \mathrm{d}r \\ &= \frac{\varepsilon_0}{2} \int E(\vec{r})^2 \,\mathrm{d}r \end{aligned}$$

Therefore, in sum:

$$W = \frac{\varepsilon_0}{2} \int E(\vec{r})^2 \,\mathrm{d}v$$

And therefore, energy density is[1]:

$$u(\vec{r}) = \frac{\varepsilon_0 E(\vec{r})^2}{2}$$

1.21. Conductors

Electrostatically, $E = 0$ inside a conductor.

Property	Insulator	Conductor
Electrons	Bound	Free
Charge motion	Very limited	Free to move
Response to electric field	Weak polar	Charge rearrangement

Table 1: Comparison of insulators and conductors

In electrostatics, free charges move until they cannot any further, which is equilibrium.

Electrostatically, $E = 0$ inside a conductor because if $E \neq 0$, the free electrons inside the conductor would move ($\vec{F} = q\vec{E}$), which cannot happen statically (no time dependence).

Since, inside a conductor, $E = 0$, there is no charge inside the conductor, by Gauss's law. This implies that all the excess charge in the conductor is at the surface of the conductor.

Further, from gauss's law, $\rho = 0$ since $\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$.

Though, this does not mean that there are no charges inside the conductor, it just means that the positive lattice ions and free electrons cancel exactly.

Since $V(\vec{a}) - V\left(\vec{b}\right) = \int_{\vec{a}}^{\vec{b}} \vec{E} \cdot \mathrm{d}\vec{\ell}$, and $\vec{E} = 0$, any two points inside a conductor are at the same electric potential (when static).

---

[1] This derivation used a ρ defined only on a finite space, but this statement generalizes, and can be directly derived.

Furthermore, since for a surface charge distribution $\vec{E}_{\parallel}^{\text{top}} = \vec{E}_{\parallel}^{\text{bottom}}$, a conductor just has a surface charge, and $\vec{E}_{\parallel}^{\text{bottom}} = 0$, then $\vec{E}_{\parallel}^{\text{top}} = 0$, and therefore the tangential component of the electric field at it's surface is zero.

1.22. Capacitor

Consider a conductor with surface charge distribution:

$$\rho_S = \Phi_S q$$

Such that $\oiint_S \Phi_S \, \mathrm{d}a = 1$. This Φ_S is unique, since the surface is at equal potential.

Thereby:

$$\int q\Phi_S \, \mathrm{d}a = q$$

Then:

$$\begin{aligned} V_c(\vec{a}) &= \int_{\infty}^{\vec{a}} \vec{E} \cdot \mathrm{d}\vec{\ell} \\ &= \frac{q}{4\pi\varepsilon_0} \oiint \frac{\Phi_S}{|\vec{a} - \vec{r}'|} \, \mathrm{d}\vec{a}' \end{aligned}$$

Then we say

$$q = CV_c$$

And therefore:

$$\frac{1}{C} = \frac{\int}{4\pi\varepsilon_0} \oiint \frac{\Phi_S}{\vec{a} - \vec{r}'} \, \mathrm{d}\vec{a}'$$

Because it is a surface of a conductor, the surface is at the same potential, so the location doesn't matter.

Consider two volumes, and move Q charge from volume 1 to volume 2, then potential between the surfaces:

$$Q = V_{12} C_{12}$$

This C_{12} depends only on the shape of the two volumes.

1.23. Parallel Plate Capacitor

Consider two parallel conductors d apart, so large that we can consider them infinitely large for the purposes of using Gauss's law (translational symmetry). Consider a charge $+Q$ on the top conductor and $-Q$ on the bottom conductor.

Since these are opposing charges, they attract, and so are on the inner plates.

$$\vec{E} = E(z)\hat{z}$$

Consider a cylinder poking into the bottom conductor.

$$\int_{\text{top}} + \cancel{\int_{\text{bottom}}}_{\text{conductor}} + \cancel{\int_{\text{sides}}}_{\text{dot product is zero}} = E(z)\Delta S$$

$$\begin{aligned} E(z)\Delta S &= \frac{\Delta S \rho_S^{\text{bot}}}{\varepsilon_0} \\ E(z) &= \frac{-Q}{A\varepsilon_0} \end{aligned}$$

Therefore

$$\vec{E} = -\frac{Q}{A\varepsilon_0}\hat{z}$$

Now compute the potential difference:

$$\begin{aligned}
V &= -\int_{\vec{a}}^{\vec{b}} \vec{E} \cdot \mathrm{d}\vec{\ell} \\
&= -\int_0^d -\frac{Q}{A\varepsilon_0}\hat{z} \cdot \hat{z}\,\mathrm{d}z \\
&= \frac{Q}{A\varepsilon_0}\int_0^d \mathrm{d}z \\
&= \frac{Q}{A\varepsilon_0}d \\
&= \frac{Qd}{A\varepsilon_0}
\end{aligned}$$

Then:

$$\frac{Q}{V} = \frac{A\varepsilon_0}{d}$$

1.24. Spherical Capacitor

Consider a conductive sphere of radius a and an outer conductive shell of radius b.

Consider a charge of $-q$ on the inner capacitor and a charge of q on the outer capacitor.

We will use Gauss's law with spherical symmetry.

$$\oiint \vec{E} \cdot \mathrm{d}\vec{a} = \frac{Q_{\text{enclosed}}}{\varepsilon_0}$$

Consider a sphere of radius r.

We need the electric field $a < r < b$ to calculate the voltage difference between the two conductors. Therefore, consider $a < r < b$:

$$\begin{aligned}
4\pi r^2 E(r) &= -\frac{q}{\varepsilon_0} \\
E(r) &= -\frac{q}{4\pi r^2 \varepsilon_0}
\end{aligned}$$

$$\begin{aligned}
V &= -\int_a^b -\frac{q}{4\pi r^2 \varepsilon_0}\hat{r} \cdot \hat{r}\,\mathrm{d}r \\
&= \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)
\end{aligned}$$

$$C = \frac{q}{V}$$

$$C = \frac{q}{\frac{q}{4\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)} = \frac{4\pi\varepsilon_0}{\frac{1}{a} - \frac{1}{b}}$$

1.25. Atom Lorentz Model

Model an electron as a fuzzy spherical cloud of radius a with uniform charge around a proton.

$$4\pi r^2 E_r(r) = \frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi a^3}\frac{-e}{\varepsilon_0}$$

$$\vec{E}(r) = \frac{-re}{4\varepsilon_0 \pi a^3}\hat{r}$$

So we can consider this as a proton-electron pair being able to be modeled as a spring with $k = \frac{e}{4\varepsilon_0 \pi a^3}$.

“Lorentz Model”.

Then, if you consider an array of these for a lattice, then when you apply the electric field to the object, the “springs” move and therefore induce a charge in the insulator.

1.26. Electric dipole

Consider two charges, one at q and one at $-q$, a distance d apart.

Set the origin at the point between them and rotate the world until they are at $r_+ = \left(0, 0, \frac{d}{2}\right)$ and $r_- = \left(0, 0, -\frac{d}{2}\right)$ respectively.

Then, we can use the formula for the potential of point charges:

$$V = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{R^+} - \frac{1}{R^-}\right)$$

where R^+ is the distance between $\vec{r}$ and $\vec{r}_+$, and R^- is the distance between $\vec{r}$ and $\vec{r}_-$.

Consider $r \gg d$:

$$\begin{aligned}
R_\pm^{-1} &= \left(r^2 \mp d\vec{r}\hat{z} + \frac{d^2}{4}\right)^{-1/2} \\
&= \frac{1}{r}\left(1 \mp \frac{d}{r}\hat{r}\hat{z} + \frac{d^2}{4r^2}\right)^{-1/2} \\
&\approx \frac{1}{r}\left(1 \mp \frac{d}{r}\hat{r}\hat{z}\right)^{-1/2} \\
&\approx \frac{1}{r}\left(1 \pm \frac{1}{2}\frac{d}{r}\hat{r}\hat{z}\right) \\
&= \frac{1}{r}\left(1 \pm \frac{d}{2r}\hat{r}\hat{z}\right)
\end{aligned}$$

Plugging this back in results in:

$$V = \frac{q}{4\pi\varepsilon_0}\frac{d\hat{r}\hat{z}}{r^2}$$

We now shall take the gradient, remembering that we are in spherical coordinates:

$$E = -\nabla V = \frac{qd}{4\pi r^3}(\hat{r}2\cos(\theta) + \hat{\varphi}\sin(\theta))$$

Let $\vec{p} = qd\hat{z}$, the dipole moment:

$$E = \frac{|p|}{4\pi r^3}(\hat{r}2\cos(\theta) + \hat{\varphi}\sin(\theta))$$

$$V = \frac{\vec{p} \cdot \hat{r}}{4\pi\varepsilon_0 r^2}$$

Notice that $\vec{p}$ points from negative to positive charge, and we can now safely rotate our coordinate system back.

In general, the dipole moment is:

$$\vec{p} = q\vec{r}$$

$\vec{r}$ in this equation is the distance between the electron and the proton when under an electric field $\vec{E}$.

The polarizability α is defined by:

$$\vec{p} = \alpha\vec{E}$$

1.27. Polarization

Let's define the polarization vector:

$$\vec{P}(\vec{r}) = \lim_{v \to 0} \frac{\sum_k \vec{p}_k}{v} \;/\!\!/\!\!/\; \frac{\mathrm{C}}{\mathrm{m}^2}$$

(this is over a material, but a material isn't actually continuous, but we shall just treat it as so, so don't go too small on your limit. The material will have dipoles because of the Lorentz model.) And therefore:

$$\vec{p} = \vec{P} \, \mathrm{d}\vec{v}$$

We can now plug this into our previous formula

$$\mathrm{d}V = \frac{1}{4\pi\varepsilon_0 r^2} \vec{p} \cdot \hat{r} \, \mathrm{d}v$$

Where $\mathrm{d}V$ is the differential voltage and $\mathrm{d}v$ is the differential volume.

Remember, let us be careful (this notation will kill everyone):

$$r = |\vec{r} - \vec{r}'|$$
$$\hat{r} = \frac{\vec{r} - \vec{r}'}{r}$$

Where $\vec{r}$ is the point to be evaluated for it's potential, and $\vec{r}'$ is the location of the volume we are considering for it's contribution to the potential. Therefore:

$$V = \frac{1}{4\pi\varepsilon_0} \int \frac{\vec{P} \cdot \hat{R}}{R^2} \, \mathrm{d}v'$$

Where

$$\vec{R} = \vec{r} - \vec{r}'$$

1.28. Effective Charge Distributions of a polarized material

$$\nabla' \left[\frac{1}{R}\right] = \frac{\hat{R}}{R^2}$$

(where ∇' indicates we are differentiating with respect to $\vec{r}'$ not $\vec{r}$).

Then:

$$V(r) = \frac{1}{4\pi\varepsilon_0} \int \vec{P}(\vec{r}') \cdot \left(\nabla' \left[\frac{1}{R}\right]\right) \mathrm{d}v'$$

$$\nabla' \cdot \left(f\vec{A}\right) = f\nabla \cdot \vec{A} + \vec{A} \cdot \nabla' f$$

Use $f = |\vec{r} - \vec{r}'|$, and $\vec{A} = \vec{P}$, which results in:

$$V = \frac{1}{4\pi\varepsilon_0}\left(\int \nabla' \cdot \frac{\vec{P}}{R}\,\mathrm{d}v' - \int \frac{\nabla' \cdot \vec{P}}{R}\,\mathrm{d}v'\right)$$

Use the divergence theorem:

$$V = \frac{1}{4\pi\varepsilon_0}\left(\oiint \frac{\vec{P} \cdot \hat{n}}{R}\,\mathrm{d}a - \int \frac{\nabla' \cdot \vec{P}}{R}\,\mathrm{d}v'\right)$$

So we can basically consider this as two charge distributions, one surface and one volume:

$$\rho_V = -\nabla \cdot \vec{P}$$
$$\rho_S = \vec{P} \cdot \hat{n}$$

Now, it is clear that if $\vec{P}$ is constant, ρ_V is zero. If $\vec{P}$ is non-constant, it also make sense that there would be a charge distribution inside the volume because this implies that the dipoles are pointing in different directions and so there is a net charge inside some particular volumes.

1.29. Electric Flux Density

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} = \frac{\rho_f + \rho_b}{\varepsilon_0}$$

ρ_b are the charges that are associated with dipoles ("bound" charges) (ρ_f are the other ones, the "free" ones).

Then $\rho_b = -\nabla \cdot \vec{P}$ and therefore:

$$\nabla \cdot \left(\varepsilon_0\vec{E} + \vec{P}\right) = \rho_f$$

Define:

$$\vec{D} = \varepsilon_0\vec{E} + \vec{P}$$

Therefore:

$$\nabla \cdot \vec{D} = \rho_f$$

And further:

$$\oiint \vec{D} \cdot \mathrm{d}\vec{a} = Q_{\text{enclosed}}$$

1.30. Linear Isotropic Homogenous Materials

In a linear isotropic homogenous material:

$$\vec{P} = \varepsilon_0\chi_e\vec{E}$$

Where χ_e is the electric susceptibility. Then:

$$\begin{aligned}\vec{D} &= \varepsilon_0\vec{E} + \vec{P} \\ &= \varepsilon_0\vec{E} + \varepsilon_0\chi_e\vec{E} \\ &= \varepsilon_0(1 + \chi_e)\vec{E}\end{aligned}$$

And therefore:

$$\vec{E} = \frac{1}{\varepsilon_0(1+\chi_e)}\vec{D}$$

Define relative permittivity, $\varepsilon_r = 1 + \chi_e$, and absolute permittivity, $\varepsilon = \varepsilon_0\varepsilon_r$:

$$\begin{aligned}\vec{E} &= \frac{1}{\varepsilon_0(1+\chi_e)}\vec{D} \\ &= \frac{1}{\varepsilon_0\varepsilon_r}\vec{D} \\ &= \frac{1}{\varepsilon}\vec{D}\end{aligned}$$

A material is called linear when the Lorentz model is valid and therefore $\vec{P} \propto \vec{E}$.

A material is isotropic when it doesn't matter the direction.

A material is homogenous when it is the same throughout.

Basically everything in this class will be such (but you do need to be told it).

2. Magnetostatics

2.1. Lorentz Force Law

Consider two wires with currents I_1, I_2, both pointing up. One could measure that when $\operatorname{sgn}(I_1) = \operatorname{sgn}(I_2)$, the wires attract, and that if $\operatorname{sgn}(I_1) \neq \operatorname{sgn}(I_2)$, the wires repel.

Now, we will give the Lorentz Force Law:

$$\vec{F} = Q\left(\vec{v} \times \vec{B}\right)$$

By the cross product, the force is orthogonal to $\vec{v}$ and $\vec{B}$, and so it is clear that the magnetic field does no work (since the vector $\vec{v}$ is in the same direction as $\mathrm{d}\vec{\ell}$).

A current is the number of charges moving through a surface per time:

$$I = \frac{\mathrm{d}Q}{\mathrm{d}t}$$

Consider a surface of surface area $\mathrm{d}s$. Define ρ, the volume charge distribution, and define $\vec{u}$ as the velocity of the charge. Then, $\mathrm{d}Q = \rho \vec{u} \cdot \hat{n} \, \mathrm{d}s \, \mathrm{d}t$, and therefore $\mathrm{d}I = \rho \vec{u} \cdot \hat{n} \, \mathrm{d}s$.

Define $\vec{J} = \rho \vec{u}$, the current density. Then:

$$\mathrm{d}I = J \cdot \mathrm{d}\vec{s}$$

Take the integral over a surface S:

$$I = \iint_S \vec{J} \cdot \mathrm{d}\vec{s}$$

Write $\vec{K} = \sigma \vec{u}$. Then, $\vec{J} = \int \vec{K} \cdot \mathrm{d}\vec{\ell}$, and $I = \lambda \vec{u}$.

Consider a volume V with ρ and $\vec{u}$, and a small piece of that called $\mathrm{d}v$.

$$q = \rho \, \mathrm{d}v$$

Therefore, by the Lorentz Force Law:

$$\mathrm{d}\vec{F} = \rho \, \mathrm{d}v\left(\vec{u} \times \vec{B}\right)$$

Which can be rewritten to:

$$\mathrm{d}\vec{F} = \vec{J} \times \vec{B} \, \mathrm{d}v$$

Taking the integral:

$$\vec{F} = \int_V \vec{J} \times \vec{B} \, \mathrm{d}v$$

For a surface S:

$$\vec{F} = \int_S \vec{K} \times \vec{B} \, \mathrm{d}s$$

and for a line L:

$$\vec{F} = \int_L \vec{I} \times \vec{B} \, \mathrm{d}\ell$$

For a line, $\vec{I} = \hat{\ell} I$, and therefore:

$$F = \int I\hat{\ell} \times B \, \mathrm{d}\ell$$

Statically, since under magnetostatics, I is constant:

$$F = I \int \mathrm{d}\vec{\ell} \times \vec{B}$$

2.2. Law of Continuity

Consider a closed surface S, and calculate the current traveling through that surface:

$$I = \oiint \vec{J} \cdot \mathrm{d}\vec{s}$$

Then:

$$\mathrm{d}Q = -I \, \mathrm{d}t$$

since we have charges flowing out of the surface (as the $\hat{n}$ in $\mathrm{d}\vec{s}$ is by convention pointing outwards). And therefore:

$$\begin{aligned} \mathrm{d}Q &= -I \, \mathrm{d}t \\ \mathrm{d}Q &= -\oiint \vec{J} \cdot \mathrm{d}\vec{s} \, \mathrm{d}t \\ -\frac{\mathrm{d}Q}{\mathrm{d}t} &= \oiint \vec{J} \cdot \mathrm{d}\vec{s} \\ -\frac{\mathrm{d}Q}{\mathrm{d}t} &= \int \nabla \cdot \vec{J} \, \mathrm{d}v \end{aligned}$$

We know that $Q = \int \rho \, \mathrm{d}v$.

$$\int \nabla \cdot \vec{J} \, \mathrm{d}v = \int -\frac{\mathrm{d}\rho}{\mathrm{d}t} \, \mathrm{d}v$$

And therefore:

$$\nabla \cdot \vec{J} = -\frac{\mathrm{d}\rho}{\mathrm{d}t}$$

Statically, charges are not changing, so:

$$\nabla \cdot \vec{J} = 0$$

And therefore:

$$0 = \int \nabla \cdot \vec{J} = \oiint \vec{J} \cdot \mathrm{d}\vec{s}$$

Break the closed surface S up into surfaces S_i, and therefore

$$0 = \oiint \vec{J} \cdot \mathrm{d}\vec{s} = \sum_i \int_{S_i} \vec{J} \cdot \mathrm{d}\vec{s}$$

Write $I_i = \int_{S_i} \vec{J} \cdot \mathrm{d}\vec{s}$, and now we have Kirchhoff's current law:

$$\sum_i I_i = 0$$

2.3. Magnetic Field from Current (Biot-Savart Law)

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}') \times \hat{R}}{R^2} \mathrm{d}v'$$

Remember that $\vec{R} = \vec{r} - \vec{r}'$. μ_0 is called the permeability of free space. In the former definition of the ampere, $\mu_0 = 4\pi \times 10^{-7}\ \mathrm{N/A^2}$.

Alternatively one can write:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} I \int \frac{\mathrm{d}\vec{\ell}' \times \hat{R}}{R^2}$$

2.4. Infinite Line with current

Consider an infinite line and a point s away from it. Rotate and translate the field until the point lands on $(0, s, 0)$ and the line is defined by all points $(x, 0, 0)$ for all x.

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} I \int \frac{\mathrm{d}\vec{\ell}' \times \hat{R}}{R^2}$$

Clearly, $\mathrm{d}\vec{\ell}' = \mathrm{d}x' \hat{x}$. Then:

$$\begin{aligned} \vec{R} &= \vec{r} - \vec{r}' \\ \vec{r} &= s\hat{y} \\ \vec{r}' &= x'\hat{x} \\ R &= \sqrt{s^2 + x'^2} \\ \hat{R} &= \frac{\vec{R}}{R} \end{aligned}$$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} I \int_{-\infty}^{\infty} \frac{\hat{x}\,\mathrm{d}x' \times (x'\hat{x} + s\hat{y})}{\left(s^2 + x'^2\right)^{3/2}}$$

$$\begin{aligned} \hat{x} \times \hat{x} &= 0 \\ \hat{x} \times \hat{y} &= \hat{z} \end{aligned}$$

And the cross product distributes!

$$\begin{aligned} \vec{B}(\vec{r}) &= \frac{\mu_0}{4\pi} I \int_{-\infty}^{\infty} \frac{s\hat{z}}{\left(s^2 + x'^2\right)^{3/2}} \\ &= \frac{\mu_0}{4\pi} I \hat{z} \int_{-\infty}^{\infty} \frac{s}{\left(s^2 + x'^2\right)^{3/2}} \\ &= \frac{\mu_0}{4\pi} I \hat{z} \frac{x'}{s\sqrt{s^2 + x'^2}} \Bigg|_{-\infty}^{\infty} \\ &= \frac{\mu_0}{4\pi} I \hat{z} \left(\frac{\infty}{s\sqrt{s^2 + \infty^2}} - \frac{-\infty}{s\sqrt{s^2 + \infty^2}} \right) \\ &= \frac{\mu_0}{4\pi} \hat{z} \left(\frac{\cancel{\infty}}{s\cancel{\sqrt{s^2 + \infty^2}}} - \frac{-\cancel{\infty}}{s\cancel{\sqrt{s^2 + \infty^2}}} \right) \\ &= \frac{\mu_0}{4\pi} I \hat{z} \left(\frac{1}{s} - \frac{-1}{s} \right) \\ &= \frac{\mu_0}{4\pi} I \hat{z} \left(\frac{2}{s} \right) \end{aligned}$$

$$= \frac{\mu_0}{4\pi} I \frac{2}{s} \hat{z}$$

$$= \frac{\mu_0 I}{2\pi s} \hat{z}$$

2.5. Divergence of $\vec{B}$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}') \times \hat{R}}{R^2} \mathrm{d}v'$$

$$\nabla \cdot \vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \left(\nabla \cdot \int \frac{\vec{J}(\vec{r}') \times \hat{R}}{R^2} \mathrm{d}v' \right)$$

$$\nabla \cdot \vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \left(\int \nabla \cdot \left(\vec{J}(\vec{r}') \times \frac{\hat{R}}{R^2} \right) \mathrm{d}v' \right)$$

$$\nabla \cdot \left(\vec{J}(\vec{r}') \times \frac{\hat{R}}{R^2} \right) = \frac{\hat{R}}{R^2} \big(\nabla \times \vec{J}(\vec{r}') \big) - \vec{J} \left(\nabla \times \frac{\hat{R}}{R^2} \right) = 0$$

This equals zero because the curl of a constant is zero and then $\nabla \times \frac{\hat{R}}{R^2} = 0$

And therefore:

$$\nabla \cdot \vec{B} = 0$$

2.6. Curl of $\vec{B}$

Begin with Biot-Savart:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}') \times \hat{R}}{R^2} \, \mathrm{d}v'$$

Take curl:

$$\begin{aligned} \nabla \times \vec{B} &= \nabla \times \left(\frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}') \times \hat{R}}{R^2} \, \mathrm{d}v' \right) \\ &= \frac{\mu_0}{4\pi} \left(\nabla \times \left(\int \frac{\vec{J}(\vec{r}') \times \hat{R}}{R^2} \, \mathrm{d}v' \right) \right) \\ &= \frac{\mu_0}{4\pi} \int \left(\nabla \times \frac{\vec{J}(\vec{r}') \times \hat{R}}{R^2} \right) \mathrm{d}v' \end{aligned}$$

Working on the subpart:

$$\nabla \times \frac{\vec{J}(\vec{r}') \times \hat{R}}{R^2}$$

$$= \nabla \times \vec{J}(\vec{r}') \times \frac{\hat{R}}{R^2}$$

$$= \vec{J} \left(\nabla \cdot \frac{\hat{R}}{R^2} \right) - \big(\vec{J} \cdot \nabla \big) \frac{\hat{R}}{R^2}$$

Note that, by textbook:

$$\int \left(\vec{J} \cdot \nabla\right) \frac{\hat{R}}{R^2} \mathrm{d}v' = 0$$

Furthermore:

$$\nabla \cdot \frac{\hat{R}}{R^2} = 4\pi\delta(\vec{r} - \vec{r}')$$

Therefore:

$$\begin{aligned}
\nabla \times \vec{B} &= \frac{\mu_0}{4\pi} \int \vec{J}(\vec{r}')(4\pi\delta(\vec{r} - \vec{r}'))\,\mathrm{d}v' \\
&= \frac{\mu_0}{4\pi} 4\pi \int \vec{J}(\vec{r}')(\delta(\vec{r} - \vec{r}'))\,\mathrm{d}v' \\
&= \mu_0 \int \vec{J}(\vec{r}')(\delta(\vec{r} - \vec{r}'))\,\mathrm{d}v' \\
&= \mu_0 \vec{J}(\vec{r})
\end{aligned}$$

And so we derive Ampere's Law:

$$\nabla \times \vec{B} = \mu_0 \vec{J}(\vec{r})$$

2.7. Ampere's Law (Integral Form) and Symmetries

Take Ampere's Law:

$$\nabla \times \vec{B} = \mu_0 \vec{J}(\vec{r})$$

Consider integrating both sides over some open surface S:

$$\begin{aligned}
\iint_S \left(\nabla \times \vec{B}\right) \cdot \mathrm{d}\vec{s} &= \iint_S \mu_0 \vec{J}(\vec{r}) \cdot \mathrm{d}\vec{s} \\
\iint_S \left(\nabla \times \vec{B}\right) \cdot \mathrm{d}\vec{s} &= \mu_0 \iint_S \vec{J}(\vec{r}) \cdot \mathrm{d}\vec{s} \\
\iint_S \left(\nabla \times \vec{B}\right) \cdot \mathrm{d}\vec{s} &= \mu_0 I_{\text{enclosed}}
\end{aligned}$$

By Stokes Theorem, consider the path P oriented according to the right hand rule for the opening on the surface S.

$$\oint_P \vec{B} \cdot \mathrm{d}\vec{\ell} = \mu_0 I_{\text{enclosed}}$$

2.7.1. Cylindrical Symmetry

Consider cylindrical symmetry, so rotating φ and moving z does not change the problem. If rotating the world doesn't change the problem, then it is impossible for B to be dependent on it. Therefore, $\vec{B}(s, \varphi, z) = \vec{B}(s)$.

Furthermore, $\vec{B}(s)$ must be perpendicular to $\vec{I}$ and $\hat{R}$ by the Biot-Savart law. Therefore, $\vec{B}(s) = B(s)\hat{\varphi}$ (since $\vec{I}$ must be in the direction of $\hat{z}$ for there to be cylindrical symmetry, by definition).

Then, consider a circular loop of radius s. We will use the surface defined by the circle with border at the loop.

Therefore:

$$\mathrm{d}\vec{\ell} = s\,\mathrm{d}\varphi\hat{\varphi}$$

Then, $\vec{B} \cdot \mathrm{d}\vec{\ell} = B(s)\hat{\varphi} \cdot s\,\mathrm{d}\varphi\hat{\varphi} = B(s)s\,\mathrm{d}\varphi$.

So:

$$
\begin{aligned}
&\int_0^{2\pi} sB(s)\,\mathrm{d}\varphi \\
&= sB(s)\int_0^{2\pi}\mathrm{d}\varphi \\
&= 2\pi sB(s)
\end{aligned}
$$

Consider the line with current I. Here, we have $I_{\text{enclosed}} = I$, and therefore:

$$
\begin{aligned}
2\pi sB(s) &= \mu_0 I \\
B(s) &= \frac{\mu_0 I}{2\pi s} \\
\vec{B} &= \frac{\mu_0 I}{2\pi s}\hat{\varphi}
\end{aligned}
$$

2.7.2. Solenoidal Symmetry

Imagine wire wrapped tightly around a cylinder, and such an angle where the direction of the current can be approximated well as $\hat{\varphi}$. We again have the fact that you can rotate about $\hat{z}$ and translate in the $\hat{z}$ direction, and therefore $\vec{B}(s,\varphi,z) = \vec{B}(s)$.

Furthermore, $\vec{B}(s) = B(s)\hat{z}$.

Consider the path that goes up a length L (segment s_2), goes to the side, then goes down (segment s_1) a length L and then goes back to the bottom of s_2. Consider this path wholly outside the loop. Then, $I_{\text{enclosed}} = 0$. The loops that go to the side do $\underbrace{\hat{z}}_{\text{direction of }\vec{B}} \cdot \underbrace{\hat{s}}_{\text{direction of }\mathrm{d}\vec{\ell}} = 0$, and so they vanish, and therefore, $(LB(s_2) - LB(s_1)) = \mu_0 I_{\text{enclosed}}$, and therefore $B(s_2) = B(s_1)$. Since at infinity, the electric field is 0, then, when wholly outside the solenoid, $B(s) = 0$.

Consider alternatively if s_1 was inside the solenoid and s_2 outside. Then, define n as the number of loops per distance, and therefore, the current through this loop is nLI. Therefore:

$$
B(s_1)L - B(s_2)L = \mu_0 nLI
$$

But since s_2 is outside, $B(s_2) = 0$. Therefore:

$$
\begin{aligned}
LB(s_1) &= \mu_0 nLI \\
B(s_1) &= \mu_0 nI
\end{aligned}
$$

2.8. Magnetic Vector Potential

Define $\vec{A}$ by $\vec{B} = \nabla \times \vec{A}$. Then, $\vec{A}$ is the magnetic vector potential.

If $\vec{B} = \nabla \times \vec{A}$ then $\vec{B} = \nabla \times \left(\vec{A} + \nabla T\right)$, as $\nabla \times \nabla T = 0$, where T is some arbitrary vector-valued function. So, if $\vec{A}$ is some magnetic vector potential, $\vec{A} + \nabla T$ is also a valid magnetic vector potential.

2.9. Poisson Equation through Coulomb Gauge

We know that

$$
\vec{E} = -\nabla V
$$

and

$$
\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}
$$

Therefore:

$$\begin{aligned}\nabla \cdot (-\nabla V) &= \frac{\rho}{\varepsilon_0} \\ \nabla \cdot \nabla V &= -\frac{\rho}{\varepsilon_0} \\ \nabla^2 V &= -\frac{\rho}{\varepsilon_0}\end{aligned}$$

This equation is called the Poisson equation.

This is solvable via:

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\vec{r}')}{R} \,\mathrm{d}v'$$

For magnetics we have:

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$

and

$$\vec{B} = \nabla \times \vec{A}$$

And the following identity:

$$\nabla \times \nabla \times \vec{A} = \nabla\left(\nabla \cdot \vec{A}\right) - \nabla^2 \vec{A}$$

And therefore:

$$\nabla \times \nabla \times \vec{A} = \mu_0 \vec{J}$$

Note that we can always choose an $\vec{A}$ such that $\nabla \cdot \vec{A} = 0$. This is called the Coulomb gauge. This then means that:

$$\begin{aligned}\nabla \times \nabla \times \vec{A} &= \mu_0 \vec{J} \\ \nabla\left(\nabla \cdot \vec{A}\right) - \nabla^2 \vec{A} &= \mu_0 \vec{J} \\ -\nabla^2 \vec{A} &= \mu_0 \vec{J} \\ \nabla^2 \vec{A} &= -\mu_0 \vec{J}\end{aligned}$$

This can be written as three separate Poisson equations:

$$\begin{aligned}\nabla^2 A_x &= -\mu_0 J_x \\ \nabla^2 A_y &= -\mu_0 J_y \\ \nabla^2 A_z &= -\mu_0 J_z\end{aligned}$$

and therefore, we can use the same solution:

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \iiint \frac{\vec{J}(\vec{r}')}{R} \mathrm{d}v'$$

2.10. Magnetic Dipoles

By the Biot-Savart law:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} I \int \frac{\mathrm{d}\vec{\ell}' \times \hat{R}}{R^2}$$

Then, consider $\vec{r}' = a\hat{\rho}$, which defines a circular loop of wire with radius a. Furthermore, notice that you can assume WLOG that the point $\vec{r}$ is on the xz-plane, as if it was not, you could rotate the problem about the z axis until it was. Call this amount of rotation φ_{obs} (to be used at the conclusion of the problem). Then, write $\vec{r} = \rho\hat{x} + z\hat{z}$.

First, expand $\hat{R}$ as $\vec{R}/R$:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} I \int \frac{\mathrm{d}\vec{\ell}' \times (\vec{r} - \vec{r}')}{R^3}$$

Then, see that we will be integrating with respect to φ' a circle of radius a, so therefore $\mathrm{d}\vec{\ell}' = a\,\mathrm{d}\varphi'\hat{\varphi}$, and the bounds will be 0 to 2π.

$$\begin{aligned}\vec{B}(\vec{r}) &= \frac{\mu_0}{4\pi} I \int_0^{2\pi} \frac{(a\,\mathrm{d}\varphi'\hat{\varphi}) \times (\vec{r} - \vec{r}')}{R^3} \\ &= \frac{\mu_0 a I}{4\pi} \int_0^{2\pi} \frac{\hat{\varphi} \times (\vec{r} - \vec{r}')}{R^3}\,\mathrm{d}\varphi'\end{aligned}$$

Then, plug in variables, and distribute the cross product:

$$\begin{aligned}&= \frac{\mu_0 a I}{4\pi} \int_0^{2\pi} \frac{\hat{\varphi} \times (\rho\hat{x} + z\hat{z} - a\hat{\rho})}{R^3}\,\mathrm{d}\varphi' \\ &= \frac{\mu_0 a I}{4\pi} \int_0^{2\pi} \frac{\rho(\hat{\varphi} \times \hat{x}) + z(\hat{\varphi} \times \hat{z}) - a(\hat{\varphi} \times \hat{\rho})}{R^3}\,\mathrm{d}\varphi'\end{aligned}$$

As $\hat{\varphi} \times \hat{\rho} = -\hat{z}$, and furthermore, $\hat{\varphi} \times \hat{z} = \hat{\rho}$. And then, for $\hat{\varphi} \times \hat{x} = (-\cos(\varphi))\hat{z}$, via $\hat{x} = \cos(\varphi)\hat{\rho} - \sin(\varphi)\hat{\varphi}$, and then performing, $\hat{\varphi} \times \hat{\rho} = -\hat{z}$ and $\hat{\varphi} \times \hat{\varphi} = 0$. Therefore:

$$\begin{aligned}&= \frac{\mu_0 a I}{4\pi} \int_0^{2\pi} \frac{\rho(-\cos(\varphi')\hat{z}) + z(\hat{\rho}) - a(-\hat{z})}{R^3}\,\mathrm{d}\varphi' \\ &= \frac{\mu_0 a I}{4\pi} \int_0^{2\pi} \frac{z\hat{\rho} + (a - \rho\cos(\varphi'))\hat{z}}{R^3}\,\mathrm{d}\varphi'\end{aligned}$$

Then, expand $\hat{\rho} = \cos(\varphi')\hat{x} + \sin(\varphi')\hat{y}$, and apply the linearity of the integral operator:

$$\begin{aligned}&= \frac{\mu_0 a I}{4\pi} \int_0^{2\pi} \frac{z(\cos(\varphi')\hat{x} + \sin(\varphi')\hat{y}) + (a - \rho\cos(\varphi'))\hat{z}}{R^3}\,\mathrm{d}\varphi' \\ &= \frac{\mu_0 a I}{4\pi} \left(\int_0^{2\pi} \frac{z\cos(\varphi')\hat{x} + z\sin(\varphi')\hat{y}}{R^3}\,\mathrm{d}\varphi' + \int_0^{2\pi} \frac{(a - \rho\cos(\varphi'))\hat{z}}{R^3}\,\mathrm{d}\varphi' \right) \\ &= \frac{\mu_0 a I}{4\pi} \left(z\left(\int_0^{2\pi} \frac{\cos(\varphi')\hat{x}}{R^3}\,\mathrm{d}\varphi' + \int_0^{2\pi} \frac{\sin(\varphi')\hat{y}}{R^3}\,\mathrm{d}\varphi' \right) + \int_0^{2\pi} \frac{(a - \rho\cos(\varphi'))\hat{z}}{R^3}\,\mathrm{d}\varphi' \right)\end{aligned}$$

Now, to compute R. First, recall $\vec{r}$ and $\vec{r}'$, and recall that $\cos(\theta)^2 + \sin(\theta)^2 = 1$:

$$\begin{aligned}R &= \left|\vec{R}\right| \\ &= |\vec{r} - \vec{r}'| \\ &= |\rho\hat{x} + z\hat{z} - a\hat{\rho}| \\ &= |\rho\hat{x} + z\hat{z} - a(\cos(\varphi')\hat{x} + \sin(\varphi')\hat{y})| \\ &= |(\rho - a\cos(\varphi'))\hat{x} + z\hat{z} - a\sin(\varphi')\hat{y}| \\ &= \sqrt{(\rho - a\cos(\varphi'))^2 + z^2 + (-a\sin(\varphi))^2}\end{aligned}$$

$$
\begin{aligned}
&= \sqrt{\rho^2 - 2a\rho\cos(\varphi') + a^2{\cos(\varphi')}^2 + z^2 + a^2{\sin(\varphi')}^2} \\
&= \sqrt{\rho^2 - 2a\rho\cos(\varphi') + a^2\left({\cos(\varphi')}^2 + {\sin(\varphi')}^2\right) + z^2} \\
&= \sqrt{\rho^2 - 2a\rho\cos(\varphi') + a^2 + z^2}
\end{aligned}
$$

Let's attack the integral with the $\hat{y}$ term.

$$
\begin{aligned}
&\int_0^{2\pi} \frac{\sin(\varphi')\hat{y}}{R^3} \,\mathrm{d}\varphi' \\
&= \hat{y}\int_0^{2\pi} \frac{\sin(\varphi')}{\left(\sqrt{\rho^2 - 2a\cos(\varphi') + a^2 + z^2}\right)^3} \,\mathrm{d}\varphi'
\end{aligned}
$$

Apply the change of variables $\theta = \varphi' - \pi$. Then, $\mathrm{d}\theta = \mathrm{d}\varphi'$, and $\sin(\varphi') = -\sin(\theta)$, and $\cos(\varphi') = -\cos(\theta)$. Further, the bounds change to $-\pi$ to π.

$$
\begin{aligned}
&= \hat{y}\int_{-\pi}^{\pi} \frac{-\sin(\theta)}{\left(\sqrt{\rho^2 + 2a\rho\cos(\theta) + a^2 + z^2}\right)^3} \,\mathrm{d}\theta \\
&= -\hat{y}\int_{-\pi}^{\pi} \frac{\sin(\theta)}{\left(\sqrt{\rho^2 + 2a\rho\cos(\theta) + a^2 + z^2}\right)^3} \,\mathrm{d}\theta
\end{aligned}
$$

Now, notice that the this a function of only θ. The denominator can be thought of as a function with an input of $\cos(\theta)$, and $\cos(\theta)$ is even, and therefore the entire denominator is even.

The numerator, $\sin(\theta)$, is odd. The product of an even function and an odd function is an odd function, and therefore, the entire term is odd. Then, for any odd function f, the symmetric integral $\int_{-y}^{y} f(x)\,\mathrm{d}x$ is zero.

Therefore, this integral evaluates to zero. Then:

$$
\begin{aligned}
\vec{B}(\vec{r}) &= \frac{\mu_0 a I}{4\pi}\left(z\left(\int_0^{2\pi} \frac{\cos(\varphi')\hat{x}}{R^3}\,\mathrm{d}\varphi' + \int_0^{2\pi} \frac{\sin(\varphi')\hat{y}}{R^3}\,\mathrm{d}\varphi'\right) + \int_0^{2\pi} \frac{(a - \rho\cos(\varphi'))\hat{z}}{R^3}\,\mathrm{d}\varphi'\right) \\
&= \frac{\mu_0 a I}{4\pi}\left(z\hat{x}\int_0^{2\pi} \frac{\cos(\varphi')}{R^3}\,\mathrm{d}\varphi' + \hat{z}\int_0^{2\pi} \frac{(a - \rho\cos(\varphi'))}{R^3}\,\mathrm{d}\varphi'\right)
\end{aligned}
$$

These remaining integrals are only expressible as elliptic integrals, so it is time to begin approximating. Let $r = \sqrt{\rho^2 + z^2}$. Since the point is far from the origin, $r \gg a$.

Consider $\frac{1}{R^3}$:

$$
\begin{aligned}
\frac{1}{R^3} &= \frac{1}{\left(\sqrt{\rho^2 - 2a\rho\cos(\varphi') + a^2 + z^2}\right)^3} \\
&= \frac{1}{\left(\sqrt{\rho^2 + z^2 - 2a\rho\cos(\varphi') + a^2}\right)^3} \\
&= \frac{1}{\left(\sqrt{r^2\left(1 - \left(\frac{2a\rho}{r^2}\right)\cos(\varphi') + \left(\frac{a}{r}\right)^2\right)}\right)^3} \\
&= \frac{1}{r^3\sqrt{1 - \left(\frac{2a\rho}{r^2}\right)\cos(\varphi') + \left(\frac{a}{r}\right)^2}^3}
\end{aligned}
$$

$$= r^{-3}\left(1-\left(\frac{2a\rho}{r^2}\right)\cos(\varphi')+\left(\frac{a}{r}\right)^2\right)^{-3/2}$$

Since $r \gg a$, $\frac{a}{r} \ll 1$, and therefore $\left(\frac{a}{r}\right)^2 \approx 0$. Furthermore, we can approximate $(1+\varepsilon)^{-3/2} \approx 1 - \frac{3}{2}\varepsilon$.

$$\begin{aligned}
&= r^{-3}\left(1+\frac{3}{2}2a\rho r^{-2}\cos(\varphi')\right)\\
&= r^{-3}(1+3a\rho r^{-2}\cos(\varphi'))
\end{aligned}$$

Substitute back into the integral, first the left part:

$$\begin{aligned}
&\int_0^{2\pi}\frac{\cos(\varphi')}{R^3}\,\mathrm{d}\varphi'\\
&= \int_0^{2\pi}\cos(\varphi')r^{-3}(1+3a\rho r^{-2}\cos(\varphi'))\,\mathrm{d}\varphi'\\
&= r^{-3}\int_0^{2\pi}\cos(\varphi')+3a\rho r^{-2}\cos(\varphi')^2\,\mathrm{d}\varphi'\\
&= r^{-3}\left(\int_0^{2\pi}\cos(\varphi')\,\mathrm{d}\varphi'+3a\rho r^{-2}\int_0^{2\pi}\cos(\varphi')^2\,\mathrm{d}\varphi'\right)\\
&= 3a\rho r^{-5}\int_0^{2\pi}\cos(\varphi')^2\,\mathrm{d}\varphi'\\
&= \frac{3a\pi\rho}{r^5}
\end{aligned}$$

Then, the right part:

$$\begin{aligned}
&\int_0^{2\pi}\frac{a-\rho\cos(\varphi')}{R^3}\,\mathrm{d}\varphi'\\
&= \int_0^{2\pi}(a-\rho\cos(\varphi'))r^{-3}(1+3a\rho r^{-2}\cos(\varphi')\,\mathrm{d}\varphi'\\
&= r^{-3}\left(\int_0^{2\pi}a\,\mathrm{d}\varphi+\int_0^{2\pi}(-\rho+a3a\rho r^{-2})\cos(\varphi')\,\mathrm{d}\varphi'+\int_0^{2\pi}-\rho 3a\rho r^{-2}\cos(\varphi')^2\,\mathrm{d}\varphi'\right)\\
&= r^{-3}\left(2\pi a+(-\rho+a3a\rho r^{-2})\cancel{\int_0^{2\pi}\cos(\varphi')\,\mathrm{d}\varphi'}-3a\rho^2r^{-2}\int_0^{2\pi}\cos(\varphi')^2\,\mathrm{d}\varphi'\right)\\
&= r^{-3}(2\pi a-3a\rho^2r^{-2}\pi)\\
&= \frac{\pi a}{r^3}\left(2-\frac{3\rho^2}{r^2}\right)\\
&= \frac{\pi a}{r^3}\left(\frac{2r^2-3\rho^2}{r^2}\right)\\
&= \frac{\pi a(2(\rho^2+z^2)-3\rho^2)}{r^5}\\
&= \frac{\pi a(2\rho^2+2z^2-3\rho^2)}{r^5}
\end{aligned}$$

$$= \frac{\pi a(2z^2 - \rho^2)}{r^5}$$

Therefore, in sum:

$$\vec{B}(\vec{r}) \approx \frac{\mu_0 a I}{4\pi}\left(z\hat{x}\frac{3a\pi\rho}{r^5} + \hat{z}\left(\frac{\pi a(2z^2 - \rho^2)}{r^5}\right)\right)$$

$$= \frac{\mu_0 a^2 I}{4r^5}(3z\rho\hat{x} + (2z^2 - \rho^2)\hat{z})$$

But, recall, we rotated the world, so we must rotate it back. Let $\hat{x} = \hat{\rho}$:

$$\vec{B}(\vec{r}) \approx \frac{\mu_0 a^2 I}{4r^5}(3z\rho\hat{\rho} + (2z^2 - \rho^2)\hat{z})$$

Define the magnetic dipole:

$$\vec{m} = IA\hat{z}$$

$$\vec{m} = I\pi b^2 \hat{Z}$$

$$\vec{A} = \frac{\mu_0 \vec{m} \times \hat{r}}{4\pi r^2}$$

$$\vec{B} = \frac{\mu_0 m}{4\pi r^3}(\hat{r}2\cos(\theta') + \hat{\varphi}\sin(\varphi'))$$

2.11. Magnetic Materials

Imagine a material composed of a bunch of tiny current loops (magnetic dipoles). Like before with polarization, define a magnetization vector:

$$\overrightarrow{M} = \lim_{\Delta v \to 0} \frac{\sum_i \vec{m}_i}{\Delta v}$$

Consider a small volume Δv. Then:

$$\vec{m} = \overrightarrow{M}\Delta v$$

$$\Delta\vec{A} = \frac{\mu_0\left(\vec{m} \times \hat{R}\right)}{4\pi R^2}$$

And then, taking it as small as possible:

$$\mathrm{d}\vec{A} = \frac{\mu_0 \overrightarrow{M} \times \hat{R}}{4\pi R^2}\,\mathrm{d}v'$$

Integrate:

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\overrightarrow{M} \times \hat{R}}{R^2}\,\mathrm{d}v'$$

Write:

$$\frac{\hat{R}}{R^2} = \nabla' \frac{1}{R}$$

Then:

$$\vec{A} = \frac{\mu_0}{4\pi}\int \overrightarrow{M} \times \left(\nabla' \frac{1}{R}\right) \mathrm{d}v'$$
$$= \frac{\mu_0}{4\pi}\int \overrightarrow{M} \times \left(\nabla' \frac{1}{R}\right) \mathrm{d}v'$$

$$\nabla' \times \frac{\overrightarrow{M}}{R} = \frac{1}{R}\nabla' \times \overrightarrow{M} - \overrightarrow{M} \times \nabla \frac{1}{R}$$

$$\overrightarrow{M} \times \nabla' \frac{1}{R} = \frac{1}{R}\nabla' \times \overrightarrow{M} - \nabla' \times \frac{\overrightarrow{M}}{R}$$

This results in:

$$\vec{A} = \frac{\mu_0}{4\pi}\left(\int \frac{\nabla' \times \overrightarrow{M}'}{R} \mathrm{d}v' + \oiint_{S'} \frac{\overrightarrow{M} \times \hat{n}}{R} \mathrm{d}s'\right)$$

So we have equivalent surface and volume current distributions:

$$\vec{J}_m = \nabla \times \overrightarrow{M}$$
$$\vec{J}_{m,s} = \overrightarrow{M} \times \hat{n}$$

2.12. The $\vec{H}$ Field

Start with ampere's law:

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$
$$\frac{1}{\mu_0}\nabla \times \vec{B} = \vec{J}$$

Write:

$$\vec{J} = \vec{J}_f + \vec{J}_m$$

In conductors, not every charge is free, only very few are being conducted.

This is also true in semiconductors, only the charges in the conduction band are moving, not those in the valence band.

Via the previous, we know that $\vec{J}_m = \nabla \times \overrightarrow{M}$ and therefore:

$$\nabla \times \underbrace{\left(\frac{\vec{B}}{\mu_0} - \overrightarrow{M}\right)}_{\vec{H}} = \vec{J}_f$$

Define the $\vec{H}$ field:

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \overrightarrow{M}$$

Therefore:

$$\nabla \times \vec{H} = \vec{J}_f$$

In the most general case, this is not entirely useful. But for specific materials (linear isotropic homogenous materials) it quite useful.

In the electric case:

$$\vec{P} = \chi_e \vec{E}$$

Similarly, magnetically:

$$\overrightarrow{M} = \chi_m \vec{H}$$

This results in:

$$\vec{B} = \mu_0 (1 + \chi_m) \vec{H}$$

Note that if $\overrightarrow{M} \propto \vec{H}$, $\overrightarrow{M} \propto \vec{B}$ by the above relation.

Define:

$$\begin{aligned} \mu_r &= 1 + \chi_m \\ \mu &= \mu_0 \mu_r \end{aligned}$$

And therefore:

$$\vec{B} = \mu \vec{H}$$

In electrical materials, χ_e is usually small[2] and always positive, but in magnetic materials, it can be classified:

Category	Sign	Size
Paramagnetic	$\chi_m > 0$	χ_m is small
Diamagnetic	$\chi_m < 0$	χ_m is small
Ferromagnetic	$\chi_m > 0$	χ_m is large

2.13. Ohm's Law

Use u for velocity.

In a broad class of materials:

$$\vec{u} = \mu \vec{E}$$

If you imagine a perfect conductor in the presence of an electric field, the electrons will just continue to accelerate (infinite velocity (though up to the speed of light)).

But in a real material, the electrons will bump into atoms in the material, which can be modeled as a frictional force proportional to velocity:

$$\begin{aligned} \vec{F} &= \frac{qE}{m} - \alpha u \\ &= m \frac{\partial u}{\partial t} \end{aligned}$$

Then, at steady state (force is zero):

$$u_{\text{SS}} = \frac{qE}{m\alpha}$$

Therefore:

$$\mu = \frac{q}{m\alpha}$$

---

[2] There are ferroelectrics but they are rare.

$$
\begin{aligned}
\vec{u} &= \mu\vec{E} \\
\vec{J} &= \rho\vec{u} \\
&= \rho\mu\vec{E} \\
&= \sigma\vec{E}
\end{aligned}
$$

We call σ the conductivity of the material.

2.13.1. Example

Consider a conductor of constant surface area s, length L with a μ and a potential difference v_0 from one end to the other.

Consider a constant electric field.

$$
\begin{aligned}
V &= -\int_0^L \vec{E} \cdot \mathrm{d}\vec{\ell} \\
&= -EL
\end{aligned}
$$

IOW:

$$E = -\frac{v_0}{L}$$

And therefore:

$$\vec{J} = -\frac{\sigma}{L} v_0$$

And by definition:

$$\vec{I} = s\vec{J}$$

And therefore:

$$
\begin{aligned}
\vec{I} &= -\frac{\sigma S}{L} v_0 \\
v_0 &= \frac{L}{S\sigma} I
\end{aligned}
$$

Define $V = IR$, therefore:

$$R = \frac{L}{S\sigma}$$

2.14. Joule's Law

Consider a particle moving under a force:

$$
\begin{aligned}
\mathrm{d}W &= \vec{F} \cdot \mathrm{d}\vec{\ell} \\
&= q\vec{E} \cdot \mathrm{d}\vec{\ell}
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
\frac{\mathrm{d}W}{\mathrm{d}t} &= q\vec{E} \cdot \frac{\mathrm{d}\vec{\ell}}{\mathrm{d}t} \\
&= q\vec{E} \cdot \vec{u}
\end{aligned}
$$

Consider this in a material, so that $q = \rho\,\mathrm{d}v$:

$$\begin{aligned}\mathrm{d}P &= \rho\,\mathrm{d}v\vec{E}\cdot\vec{u} \\ &= \vec{J}\cdot\vec{E}\,\mathrm{d}v\end{aligned}$$

Integrate:

$$P = \int \vec{J}\cdot\vec{E}\,\mathrm{d}v$$

P is the power dissipated.

Once again consider the conductor of constant surface area s, length L.

Since the electric field is in the direction of the magnetic field:

$$\vec{J}\cdot\vec{E} = JE$$

$$P = JE\int \mathrm{d}v = JEsL$$

We have $Js = I$, and $EL = V$, and therefore:

$$P = IV$$

For ohmic materials (resistors):

$$P = I^2R$$

2.15. Motional Electromotive Force (EMF)

$W = -\oint \vec{F}\cdot\mathrm{d}\vec{\ell}$

$$\begin{aligned}\vec{F} &= q\left(\vec{u}\times\vec{b}\right) \\ &= quB\hat{y}\end{aligned}$$

Therefore:

$$W = quBh$$

$$\frac{W}{q} = \mathcal{E} = \mathrm{EMF}$$

Therefore:

$$\mathcal{E} = vBh$$

And

$$I = \frac{\mathcal{E}}{R}$$

We can define the magnetic flux:

$$\begin{aligned}\Phi &= \int \vec{B}\cdot\mathrm{d}\vec{a} \\ &= Bhx\end{aligned}$$

Then, take the derivative:

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = Bh\frac{\mathrm{d}x}{\mathrm{d}t} = -Bhv = -\mathcal{E}$$

We now have the Flux Rule:

$$\mathcal{E} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t}$$

This applies in general (despite the fact that this was a derivation only for a square loop).

Consider the inertial reference frame of the loop. Then, you can consider this as the magnetic field moving with velocity v in the opposite direction. Note that this means that the magnetic field is no longer constant with time, so we are *not* in magnetostatics.

If you attempt to analyze the situation statically, you will get that the current is 0, which is incorrect.

2.16. Faraday's Law

Since we found that by the above derivation that $\mathcal{E} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t}$, and then took the inertial reference frame, resulting in a dynamic situation, we can get the differential form of Faraday's law:

$$\mathcal{E} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t}$$

Therefore, by the definition of electric potential:

$$\oint \vec{E} \cdot \mathrm{d}\vec{\ell} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t}$$

Take also:

$$\begin{aligned} \Phi &= \int \vec{B} \cdot \mathrm{d}\vec{a} \\ \to \frac{\mathrm{d}\Phi}{\mathrm{d}t} &= \int \frac{\partial \vec{B}}{\partial t} \,\mathrm{d}\vec{a} \end{aligned}$$

Therefore:

$$\begin{aligned} \oint \vec{E} \cdot \mathrm{d}\vec{\ell} &= -\int \frac{\partial \vec{B}}{\partial t} \,\mathrm{d}\vec{a} \\ \int \nabla \times \vec{E} \cdot \mathrm{d}\vec{a} &= -\int \frac{\partial \vec{B}}{\partial t} \,\mathrm{d}\vec{a} \end{aligned}$$

And we get a new version of Faraday's law:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

This also exists

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$

2.16.1. Example

Consider a infinitely long cylinder of radius a with time-varying magnetic field $\vec{B}(t)$ pointing in direction of the cylinder.

Write $\vec{E} = E(s)\hat{\varphi}$, considering an "Amperian" loop of radius s.

Use the formula

$$\oint \vec{E} \cdot \mathrm{d}\vec{\ell}$$

$$= E(2\pi s)$$

$$= -\frac{\partial \Phi}{\partial t}$$

Note that $\Phi = \pi a^2 B(t)$, and therefore:

$$\begin{aligned} E &= -\frac{1}{2\pi s}\left(\pi a^2 \frac{\partial B}{\partial t}\right) \\ &= -\frac{a^2}{2s}\frac{\partial B}{\partial t} \end{aligned}$$

2.17. Lenz's Law

The induced current always opposes the change in magnetic field.

Applying to the previous example, the electric field is in the clockwise direction (assuming that $\vec{B}$ is increasing with time).

This is often the best way to get the direction of the electric field.

2.18. Inductors

Consider two loops in space. Run a current I_1 in loop 1. Consider the flux Φ_{12}, the flux through loop 2 induced by loop 1′s current.

$$\Phi_{12} = \int_S \vec{B}_1 \cdot \mathrm{d}\vec{a}$$

Using the Biot-Savart law:

$$\vec{B}_1 = \frac{\mu_0 I_1}{4\pi} \oint_{L_1} \frac{\mathrm{d}\vec{\ell}' \times \hat{R}}{R^2}$$

$$\Phi_{12} = \frac{\mu_0 I_1}{4\pi} \int_{S_2} \oint \frac{\mathrm{d}\vec{\ell}' \times \hat{R}}{R^2} \cdot \mathrm{d}\vec{a}$$

See that the value $\vec{B}_1$ is just a constant, and therefore:

$$\Phi_{12} = L_{12} I_1$$

Define flux linkage, via the number of loops N_2:

$$\Lambda_{12} = N_2 \Phi_{12}$$

And then inductance is:

$$L_{12} = \frac{\Lambda_{12}}{I_1}$$

The above is the typically called the "mutual" inductance.

There is also the "self" inductance:

$$L_{11} = \frac{\Lambda_{11}}{I_1}$$

2.18.1. Inductance of a solenoid

Consider a long solenoid of length k with surface area S with N tightly wound loops.

First, recall the magnetic field for a solenoid is

$$\vec{B} = \mu_0 n I \hat{z}$$

Second, calculate the flux through one loop:

$$\Phi = BS = \mu_0 n I S$$

Third, calculate the flux linkage:

$$\Lambda = \Phi N$$

Fourth, calculate L:

$$L = \frac{\Lambda}{I} = \frac{\mu_0 n I S N}{I} = \mu_o n S N = \mu_0 n^2 k S$$

2.19. Symmetry of mutual inductance

Recall:

$$L_{12} = \frac{N_2}{I_1} \int_{S_2} \vec{B}_1 \cdot \mathrm{d}\vec{a}$$

Now, write:

$$\vec{B}_1 = \nabla \times \vec{A}_1$$

Using Stoke's Theorem:

$$\begin{aligned} L_{12} &= \frac{N_2}{I_1} \int_{S_2} \left(\nabla \times \vec{A}_1\right) \cdot \mathrm{d}\vec{a} \\ &= \frac{N_2}{I_1} \oint_{C_2} \vec{A}_1 \cdot \mathrm{d}\vec{\ell} \end{aligned}$$

Furthermore, we have the formula:

$$\vec{A}_1 = \frac{\mu_0 N_1 I_1}{4\pi} \oint_{C_1} \frac{\mathrm{d}\vec{\ell}}{R}$$

Then, we find the Neumann formula:

$$L_{12} = \frac{\mu_0 N_1 N_2}{4\pi} \oint_{C_1} \oint_{C_2} \frac{\mathrm{d}\vec{\ell}_1 \cdot \mathrm{d}\vec{\ell}_2}{R}$$

This shows that $L_{12} = L_{21}$.

2.19.1. Example

Consider a not infinitely long solenoid 1 around a second infinitely long solenoid. This is an example of why $L_{12} = L_{21}$ is convenient: computing L_{12} is hard, but L_{21} is easy.

2.20. Magnetic Induction

Recall:

$$\mathcal{E} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t}$$

And then, $\Phi = L_{11} I_1$, and therefore:

$$\mathcal{E} = -L_{11}\frac{\mathrm{d}I_1}{\mathrm{d}t}$$

Similarly, with two loops:

$$\Phi_2 = L_{21}I_1$$

$$\mathcal{E} = -L_{12}\frac{\mathrm{d}I_1}{\mathrm{d}t}$$

2.21. Work

Consider $P = IV$. Therefore:

$$P = \mathcal{E}I = L\frac{\mathrm{d}I}{\mathrm{d}t}I$$

Therefore, since $P = \frac{\mathrm{d}W}{\mathrm{d}t}$:

$$\begin{aligned}
\int P \,\mathrm{d}t &= \int \frac{\mathrm{d}W}{\mathrm{d}t}\,\mathrm{d}t \\
\int L\frac{\mathrm{d}I}{\mathrm{d}t}I\,\mathrm{d}t &= \int \mathrm{d}W \\
L\int \frac{\mathrm{d}I}{\mathrm{d}t}I\,\mathrm{d}t &= W \\
L\int I\,\mathrm{d}I &= W \\
L\frac{I^2}{2} &= W \\
W &= \frac{1}{2}LI^2
\end{aligned}$$

2.22. Example

Consider two loops. First, increase the current in loop 1 to some current I_1 from current 0, and then increase the current in loop 2 to some current I_2 from current 0.

Then, the total work is

$$W = \frac{1}{2}L_{11}I_1^2 + \frac{1}{2}L_{22}I_2^2 + L_{21}I_1I_2$$

This can be generalized to N loops, each carrying current I_j:

$$W = \frac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{N}L_{jk}I_jI_k$$

Volume formula:

$$W = \frac{1}{2}\int\left(\vec{A}\cdot\vec{J}\right)\mathrm{d}v$$

Use the fact that $\vec{J} = \frac{1}{\mu_0}\nabla\times\vec{B}$:

$$W = \frac{1}{2}\int\left(\vec{A}\cdot\left(\frac{1}{\mu_0}\nabla\times\vec{B}\right)\right)\mathrm{d}v$$

Further, use:

$$\vec{A}\cdot\left(\nabla\times\vec{B}\right)=\vec{B}\cdot\vec{B}-\nabla\cdot\left(\vec{A}\times\vec{B}\right)$$

$$\begin{aligned}W&=\frac{1}{2\mu_0}\left(\int B^2\,\mathrm{d}v-\int\left(\nabla\cdot\left(\vec{A}\times\vec{B}\right)\mathrm{d}v\right)\right)\\&=\frac{1}{2\mu_0}\left(\int B^2\,\mathrm{d}v-\oiint_S\left(\vec{A}\times\vec{B}\right)\mathrm{d}\vec{a}\right)\end{aligned}$$

Pick a sphere approaching a radius of infinity.

Since $\vec{A}\overset{\propto}{\longrightarrow}\frac{1}{r}$, $\vec{B}\overset{\propto}{\longrightarrow}\frac{1}{r^2}$ and $a\overset{\propto}{\longrightarrow}r^2$,

$$\oiint_S\vec{A}\times\vec{B}\,\mathrm{d}\vec{a}\to0$$

Therefore:

$$W=\frac{1}{2\mu_0}\int B^2\,\mathrm{d}v$$

We can use this to find the inductance of something since $W=\frac{1}{2}LI^2$.

2.23. Example

Consider a coaxial cable with solid inner core of radius a with a uniform current density and a outer shell (infinitely thin) of radius b. Find the inductance per unit length.

Consider the current on the inner core to be I and the current on the outer core to be I in the opposite direction.

Then, for the inner core, $\vec{J}=\frac{I}{\pi a^2}$.

Consider a Amperian loop of radius r.

First consider $0<r<a$. Therefore, $I_{\text{enclosed}}=Ir^2/a^2$. Using ampere's law and cylindrical symmetry:

$$\begin{aligned}B\cdot2\pi r&=\frac{\mu_0Ir^2}{a^2}\\B&=\frac{\mu_0Ir^2}{2\pi a^2r}=\frac{\mu_0Ir}{2\pi a^2}\end{aligned}$$

Add the direction, $\hat{\varphi}$.

$$\vec{B}=\frac{\mu_0Ir}{2\pi a^2}\hat{\varphi}$$

For $a<r<b$, $I_{\text{enclosed}}=I$.

$$B=\frac{\mu_0I}{2\pi r}$$

Add the direction, $\hat{\varphi}$.

$$\vec{B}=\frac{\mu_0I}{2\pi r}\hat{\varphi}$$

For $b<r$, $I_{\text{enclosed}}=0$, and therefore $\vec{B}=0$.

Consider finding the energy over some cylinder of length ℓ. Now find work:

$$W=\frac{1}{2\mu_0}\int B^2\,\mathrm{d}V$$

$$= \frac{1}{2\mu_0} \int_0^{2\pi} \int_0^{\ell} \int_0^{\infty} B^2 p \, \mathrm{d}\rho \, \mathrm{d}z \, \mathrm{d}\varphi$$

Since for $\rho > b$, $\vec{B} = 0$, we can safely ignore it.

$$W = \frac{1}{2\mu_0} (2\pi)(\ell) \int_0^b B^2 \rho \, \mathrm{d}\rho$$

Furthermore we can split the integral into the two parts from 0 to a and from a to b:

$$\begin{aligned}
W &= \frac{1}{2\mu_0} (2\pi)(\ell) \left(\int_0^a \left(\frac{\mu_0 I \rho}{2\pi a^2} \right)^2 \rho \, \mathrm{d}\rho + \int_a^b \left(\frac{\mu_0 I}{2\pi \rho} \right)^2 \rho \, \mathrm{d}\rho \right) \\
&= \frac{\mu_0 \ell I^2}{4\pi} \left(\int_0^a \left(\frac{\rho}{a^2} \right)^2 \rho \, \mathrm{d}\rho + \int_a^b \left(\frac{1}{\rho} \right)^2 \rho \, \mathrm{d}\rho \right) \\
&= \frac{\mu_0 \ell I^2}{4\pi} \left(\frac{1}{a^4} \left(\frac{a^4}{4} \right) + \ln(b) - \ln(a) \right) \\
&= \frac{\mu_0 \ell I^2}{4\pi} \left(\frac{1}{4} + \ln\left(\frac{b}{a} \right) \right)
\end{aligned}$$

Then, using $W = \frac{1}{2} L I^2$, solve for inductance, $2W / I^2 = L$:

$$L = \frac{\mu_0 \ell}{2\pi} \left(\frac{1}{4} + \ln\left(\frac{b}{a} \right) \right)$$

2.24. Laws of Electricity and Magnetism

Gauss's Law:

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

Unnamed Law:

$$\nabla \cdot \vec{B} = 0$$

Faraday's Law:

$$\nabla \cdot \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Ampere's Law:

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$

Law of Continuity:

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$$

2.25. Contradiction

Write:

$$\begin{aligned}
\nabla \cdot \left(\nabla \times \vec{B} \right) &= \nabla \cdot \left(\mu_0 \vec{J} \right) \\
\nabla \cdot \left(\nabla \times \vec{B} \right) &= \mu_0 \left(\nabla \cdot \vec{J} \right)
\end{aligned}$$

$$0 = \mu_0 \left(-\frac{\partial \rho}{\partial t} \right)$$

So this is not necessarily true. The suspicious law is Ampere’s law, as the law of continuity only relies on the conservation of mass, and Ampere’s law relied on magnetostatics.

Start with the continuity equation:

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$$

$$\nabla \cdot \vec{J} = -\frac{\partial}{\partial t}\big(\varepsilon_0 \big(\nabla \cdot \vec{E}\big)\big)$$

$$\nabla \cdot \vec{J} = -\nabla \cdot \left(\varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right)$$

$$\nabla \cdot \left(\vec{J} - \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right) = 0$$

3. Electrodynamics

3.1. Maxwell's Equations

Gauss's Law:

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

Gauss's Law for Magnetism:

$$\nabla \cdot \vec{B} = 0$$

Faraday's Law:

$$\nabla \cdot \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Ampere-Maxwell Law:

$$\nabla \times \vec{B} = \mu_0 \left(\vec{J} - \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right)$$

These laws are correct both in the static case *and* in the dynamic case.

Now, we also have the force on charges and currents:

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

3.2. In matter

We define the bound distributions:

$$\rho_b = -\nabla \cdot \vec{P}$$
$$\vec{J}_b = \nabla \times \overrightarrow{M}$$

And furthermore, take the derivative of first with respect to time:

$$\nabla \cdot \frac{\partial \vec{P}}{\partial t} = -\frac{\partial \rho_b}{\partial t}$$

And therefore, we can, by the continuity equation, get:

$$\frac{\partial \vec{P}}{\partial t} = \vec{J}_p$$

Therefore

$$\rho = \rho_f + \rho_b = \rho_f - \nabla \cdot \vec{P}$$

$$\vec{J} = \vec{J}_f + \vec{J}_b + \vec{J}_p = \vec{J}_f + \nabla \times \overrightarrow{M} + \frac{\partial \vec{P}}{\partial t}$$

We define:

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$$
$$\vec{H} = \frac{1}{\mu_0} \vec{B} - \overrightarrow{M}$$

And then we can write Maxwell's equations in matter:

$$\nabla \cdot \vec{D} = \rho_f$$
$$\nabla \cdot \vec{B} = 0$$
$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
$$\nabla \times \vec{H} = \vec{J}_f + \frac{\nabla \vec{D}}{\partial t}$$

In the case where it is a linear isotropic homogenous material:

$$\vec{D} = \varepsilon \vec{E}$$
$$\vec{H} = \frac{1}{\mu} \vec{B}$$

Then, we can simplify Maxwell's equations in matter:

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon}$$
$$\nabla \cdot \vec{B} = 0$$
$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
$$\nabla \times \vec{B} = \mu \left(\vec{J} + \varepsilon \frac{\partial \vec{E}}{\partial t} \right)$$

3.3. Electromagnetic Waves

The single-dimensional wave equation:

$$\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}$$

Where $f(z, t)$ is some function. The solutions are:

$$f(z, t) = g(z - vt)$$

We can specifically consider sinusoidal waves, since, by the fourier transform, it can be extended, because the wave equation is linear and therefore you can apply the superposition property:

$$f(z, t) = A \cos(k(z - vt) + \delta)$$

Consider specifically $\omega = kv$:

$$f(z, t) = A \cos(kz - \omega t + \delta)$$

Then, the wavelength of this wave is peak-to-peak distance, $\lambda = 2\pi / k$, and furthermore, $\omega = 2\pi f$, where f is the frequency.

We can also express this using complex exponentials:

$$\begin{aligned} f(z, t) &= \mathrm{Re}\left\{ A e^{i(kz - \omega t + \delta)} \right\} \\ &= \mathrm{Re}\left\{ A e^{i\delta} e^{i(kz - \omega t)} \right\} \end{aligned}$$

And then, $\tilde{f}(z, t) = \tilde{A} e^{i(kz - \omega t)}$, where $\tilde{A} = A e^{i\delta}$. It is often convenient to take the real part at the end, which you can do since the equation is linear.

The wave equation in 3D is:

$$\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}\right)=\frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

$$\nabla^2 f=\frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

(The Laplacian).

Then the solution is:

$$f(\vec{r},t)=g\left(\hat{k}\cdot\vec{r}-vt\right)$$

For the sinusoidal case:

$$\cos\left(k\left(\hat{k}\cdot\vec{r}-vt\right)\right)$$

$$=\cos\left(\vec{k}\cdot\vec{r}-\omega t\right)$$

And:

$$\tilde{f}(\vec{r},t)=\tilde{A}e^{i\left(\vec{k}\cdot\vec{r}-\omega t\right)}$$

In vacuum, we have $\rho=0$ and $\vec{J}=0$ and therefore:

$$\nabla\cdot\vec{E}=0$$

$$\nabla\cdot\vec{B}=0$$

$$\nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$

$$\nabla\times\vec{B}=\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$$

Then:

$$\nabla\times\nabla\times\vec{E}$$

$$=-\frac{\partial}{\partial t}\left[\nabla\times\vec{B}\right]$$

$$=-\mu_0\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2}$$

Mathematically:

$$\nabla\times\nabla\vec{E}=\nabla\left(\nabla\cdot\vec{E}\right)-\nabla^2\vec{E}$$

But we know $\nabla\cdot\vec{E}=0$, and therefore $\nabla\times\nabla\vec{E}=-\nabla^2\vec{E}$, so in sum:

$$\nabla^2\vec{E}=\mu_0\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2}$$

The same derivation can be done for magnetic waves:

$$\nabla^2\vec{B}=\mu_0\varepsilon_0\frac{\partial^2\vec{B}}{\partial t^2}$$

From this, the velocity is:

$$v=\frac{1}{\sqrt{\mu_0\varepsilon_0}}$$

We can measure these values, and therefore:

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 3 \times 10^8 \text{ m/s}$$

3.4. Monochromatic Plane Wave

Monochromatic means single frequency, plane waves are only solutions that are constant perpendicular to the direction of propagation ($\vec{k}$):

$$\vec{E}(\vec{r}, t) = \vec{E}_0 \exp\Big(i\Big(\vec{k} \cdot \vec{r} - \omega t\Big)\Big)$$
$$\vec{B}(\vec{r}, t) = \vec{B}_0 \exp\Big(i\Big(\vec{k} \cdot \vec{r} - \omega t\Big)\Big)$$

Via previous, we know that $\omega / c = k$. Therefore, the only free parameters are direction (*not* magnitude) and frequency.

These are not necessarily solutions to Maxwell's equations, we only know that every solution to Maxwell's equations is a solution to the wave equation.

First, apply Gauss's law (for electric and magnetism):

$$\nabla \cdot \vec{E} = 0 \Rightarrow \vec{k} \cdot \vec{E} = 0$$
$$\nabla \cdot \vec{B} = 0 \Rightarrow \vec{k} \cdot \vec{B} = 0$$

Therefore, the direction of the field is orthogonal to the direction of propagation.

$$\nabla \times \vec{E} = \nabla \times \Big(E_0 \exp\Big(i\Big(\vec{k} \cdot \vec{r} - \omega t\Big)\Big)\Big)$$
$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
$$\vec{B}_0 = \frac{1}{\omega}\Big(\vec{k} \times \vec{E}_0\Big)$$

Therefore, the magnetic field must be orthogonal to the electric field.

Furthermore:

$$\left|\vec{B}_0\right| = \frac{1}{c}\left|\vec{E}_0\right|$$

Consider if $\vec{E}_0 = E_0 \hat{x}$, and $\vec{k} = k\hat{z}$, then this directly implies that $\vec{B}_0 = (E_0/c)\hat{y}$.

This applies in a vacuum, and applies to all sorts of transmissions: gamma waves, microwaves, ultraviolet, visible light, infrared, radio waves, etc.

A very similar derivation can be done in a material, and you get a very similar result. The wave equations still apply, and they remain orthogonal. The velocity is now:

$$v = \frac{1}{\sqrt{\mu \varepsilon}}$$

This can sometimes have $v > c$. This does not mean that information travels faster than light, just that the phase velocity is greater than c, which is fine.

Write:

$$n = \sqrt{\frac{\varepsilon \mu}{\varepsilon_0 \mu_0}} = \frac{c}{v}$$

This is the index of refraction of the material.

3.5. Poynting's Theorem

The continuity equation is:

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = -\oiint_S \vec{J} \cdot \mathrm{d}\vec{a}$$

Where $Q = \int_V \rho(\vec{r}, t)\,\mathrm{d}v$.

The energy in an electric field is:

$$W_e = \frac{\varepsilon_0}{2} \int \left|\vec{E}\right|^2 \mathrm{d}v$$

In a magnetic field:

$$W_b = \frac{1}{2\mu_0} \int \left|\vec{B}\right|^2 \mathrm{d}v$$

Therefore the total energy is $U_{\mathrm{em}} = W_e + W_b$, the sum of the energy in the electric and magnetic fields.

The work on a small particle is:

$$\mathrm{d}w = \vec{F} \cdot \mathrm{d}\vec{\ell}$$

and

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

where $\mathrm{d}\vec{\ell} = \vec{v}\,\mathrm{d}t$.

Then:

$$\begin{aligned} \mathrm{d}w &= q\left(\vec{E} + \vec{v} \times \vec{B}\right) \cdot \vec{v}\,\mathrm{d}t \\ \mathrm{d}w &= q\vec{E} \cdot \vec{v}\,\mathrm{d}t \end{aligned}$$

Notice that this means that magnetic fields do not do work.

Write $q = \rho\,\mathrm{d}v$. Then:

$$\begin{aligned} \mathrm{d}w &= \int \rho\vec{E} \cdot \vec{v}\,\mathrm{d}v\,\mathrm{d}t \\ \frac{\mathrm{d}w}{\mathrm{d}t} &= \int \rho\vec{E} \cdot \vec{v}\,\mathrm{d}v \\ \frac{\mathrm{d}w}{\mathrm{d}t} &= \int \vec{E} \cdot \vec{J}\,\mathrm{d}v \end{aligned}$$

Recall that:

$$\nabla \times \vec{B} = \mu_0\vec{J} + \varepsilon_0\mu_0\frac{\partial\vec{E}}{\partial t}$$

Write:

$$\vec{J} = \frac{1}{\mu_0}\left(\nabla \times \vec{B}\right) - \varepsilon_0\frac{\partial\vec{E}}{\partial t}$$

Then:

$$\begin{aligned}\frac{\mathrm{d}w}{\mathrm{d}t} &= \int \vec{E} \cdot \left(\frac{1}{\mu_0}\left(\nabla \times \vec{B}\right) - \varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right) \mathrm{d}v \\ &= \frac{1}{\mu_0} \int \vec{E} \cdot \left(\nabla \times \vec{B}\right) \mathrm{d}v - \varepsilon_0 \int \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} \mathrm{d}v \\ &= -\frac{\mathrm{d}}{\mathrm{d}t}\left[\int \frac{1}{2}\left(\varepsilon_0 \left|\vec{E}\right|^2 + \frac{1}{\mu_0}\left|\vec{B}\right|^2\right) \mathrm{d}v\right] - \frac{1}{\mu_0} \oiint \left(\vec{E} \times \vec{B}\right) \cdot \mathrm{d}\vec{a} \\ &= -\frac{\mathrm{d}}{\mathrm{d}t}[U_{\mathrm{em}}] - \frac{1}{\mu_0} \oiint \left(\vec{E} \times \vec{B}\right) \cdot \mathrm{d}\vec{a}\end{aligned}$$

Define:

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}$$

Then:

$$\frac{\mathrm{d}w}{\mathrm{d}t} = -\frac{\mathrm{d}U_{\mathrm{em}}}{\mathrm{d}t} - \oiint \vec{S} \cdot \mathrm{d}\vec{a}$$

And therefore:

$$\frac{\mathrm{d}U_{\mathrm{em}}}{\mathrm{d}t} = -\frac{\mathrm{d}w}{\mathrm{d}t} - \oiint \vec{S} \cdot \mathrm{d}\vec{a}$$

Therefore, there are two ways that the energy could change. Either, the particles did work on something ($\frac{\mathrm{d}w}{\mathrm{d}t}$), or the energy flowed out the volume ($\oiint \vec{S} \cdot \mathrm{d}\vec{a}$).

3.5.1. Example: Monochromatic Plane Wave

Let:

$$\vec{E} = E_0 \cos(kz - \omega t)\hat{x}$$

Therefore:

$$\vec{B} = \frac{E_0}{c} \cos(bz - \omega t)\hat{y}$$

Define:

$$U = \int u \, \mathrm{d}v$$

Where we have energy density:

$$u = \frac{\varepsilon_0 \left|\vec{E}\right|^2}{2} + \frac{\left|\vec{B}\right|^2}{2\mu_0}$$

Insert values:

$$\begin{aligned}u &= \frac{1}{2} E_0^2 \cos^2(kz - \omega t)\left(\varepsilon_0 + \frac{1}{c^2 \mu_0}\right) \\ &= \frac{1}{2} E_0^2 \cos^2(kz - \omega t)(\varepsilon_0 + \varepsilon_0) \\ &= \frac{1}{2} E_0^2 \cos^2(kz - \omega t)(\varepsilon_0 + \varepsilon_0)\end{aligned}$$

$$= \varepsilon_0 E_0^2 \cos^2(kz - \omega t)$$

What about the Poynting vector?

$$\begin{aligned} \vec{S} &= \frac{1}{\mu_0} \vec{E} \times \vec{B} \\ &= \frac{1}{c\mu_0} E_0^2 \cos^2(kz - \omega t)\hat{z} \\ &= c\varepsilon_0 E_0^2 \cos^2(kz - \omega t)\hat{z} \\ &= cu\hat{z} \end{aligned}$$

So this shows that the energy is flowing in the direction of the propagation at the speed of light.

Consider now the time-averaged Poynting vector, denoted $\left\langle \vec{S} \right\rangle$. Then, since $\cos^2(\theta) = (1 + \cos(2\theta))/2$, it is clear the average of the cosine is $1/2$. Then:

$$\left\langle \vec{S} \right\rangle = \frac{1}{2} c\varepsilon_0 E_0^2 \hat{z}$$

Sometimes this is denoted I, the intensity. This is in watts per square meter.

3.6. Interface between materials

Consider a planar interface on the xy-plane, where towards $-\hat{z}$, the permittivity is ε_1, and the permeability is μ_1, and towards $+\hat{z}$, the permittivity is ε_2, and the permeability is μ_2.

Consider a propagating wave normal to the plane with an associated $\vec{K}_I$ and $\vec{E}_I$ (I for incidence) becoming a wave with a $\vec{K}_T$ and $\vec{E}_T$ (T for transmitted).

Then, we will use the boundary conditions:

$$\begin{aligned} E_-^\parallel &= E_+^\parallel \\ \frac{1}{\mu_1} B_-^\parallel &= \frac{1}{\mu_2} B_+^\parallel \end{aligned}$$

There will also be a field $\vec{K}_R$ and $\vec{E}_R$, (R for reflected).

Define

$$\begin{aligned} v_1 &= \frac{1}{\sqrt{\mu_1 \varepsilon_1}} \\ v_2 &= \frac{1}{\sqrt{\mu_2 \varepsilon_2}} \end{aligned}$$

$$\begin{aligned} \vec{E}_I &= E_{I0} e^{i(k_I z - \omega t)} \hat{x} \\ \vec{B}_I &= \frac{E_{I0}}{v_1} e^{i(k_I z - \omega t)} \hat{y} \end{aligned}$$

$$\begin{aligned} \vec{E}_R &= E_{R0} e^{i(-k_I z - \omega t)} \hat{x} \\ \vec{B}_R &= -\frac{E_{R0}}{v_1} e^{i(-k_I z - \omega t)} \hat{y} \end{aligned}$$

$$\begin{aligned} \vec{E}_T &= E_{T0} e^{i(k_T z - \omega t)} \hat{x} \\ \vec{B}_T &= \frac{E_{T0}}{v_2} e^{i(k_T z - \omega t)} \hat{y} \end{aligned}$$

Recall that:

$$k = \omega/v$$

So the only unknowns are E_{I0}, E_{R0}, E_{T0}. We will consider E_{I0} as a known.

Then:

$$\begin{aligned}\vec{E}_- &= \vec{E}_I + \vec{E}_R \\ \vec{B}_- &= \vec{B}_I + \vec{B}_R \\ \vec{E}_+ &= \vec{E}_T \\ \vec{B}_+ &= \vec{B}_T\end{aligned}$$

Therefore:

$$E_-^{\parallel} = E_+^{\parallel} \Rightarrow E_- = E_+$$

$$E_{I0} + E_{R0} = E_{T0}$$

Furthermore:

$$\frac{1}{\mu_1} B_-^{\parallel} = \frac{1}{\mu_2} B_+^{\parallel} \Rightarrow B_- = B_+$$

$$\begin{aligned}\frac{1}{\mu_1}(B_{I0} + B_{R0}) &= \frac{1}{\mu_2} B_{T0} \\ \Rightarrow \frac{1}{\mu_1 v_1}(E_{I0} - E_{R0}) &= \frac{1}{\mu_2 v_2} E_{T0}\end{aligned}$$

So we have the equations:

$$\begin{aligned}\frac{1}{\mu_1 v_1}(E_{I0} - E_{R0}) &= \frac{1}{\mu_2 v_2} E_{T0} \\ E_{I0} + E_{R0} &= E_{T0}\end{aligned}$$

These can be solved. We will do so by defining

$$\beta = \frac{\mu_1 v_1}{\mu_2 v_2}$$

And then:

$$E_{R0} = \frac{1-\beta}{1+\beta} E_{I0}$$

$$E_{T0} = \frac{2}{1+\beta} E_{I0}$$

3.7. Intensity

For a plane wave:

$$I = \frac{1}{2} \varepsilon v E_0^2$$

So an for an incident wave:

$$I_I = \frac{1}{2} \varepsilon_1 v_1 E_0^2$$

Then, the reflected and transmitted intensities are:

$$\begin{aligned} I_R &= \frac{1}{2}\varepsilon_1 v_1 E_{R0}^2 \\ I_T &= \frac{1}{2}\varepsilon_2 v_2 E_{T0}^2 \end{aligned}$$

Consider the special case where $\mu_1 = \mu_2$.[3]

Then:

$$\begin{aligned} R &= \frac{I_R}{I_I} = \left(\frac{v_1 - v_2}{v_1 + v_2}\right)^2 \\ T &= \frac{I_T}{I_1} \quad = \frac{2v_1 v_2}{(v_1 + v_2)^2} \end{aligned}$$

Notice that $R + T = 1$.

We can simplify further, where $n = \frac{c}{v}$:

$$\begin{aligned} R &= \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2 \\ T &= \frac{4n_1 n_2}{(n_1 + n_2)^2} \end{aligned}$$

3.8. Reflectance in practical situations

Consider the fact that for air $n = 1$, so for glass, where $n = 1.5$:

$$R = 4\%, T = 96\%$$

For silicon, where $n = 3.8$:

$$R = 40\%, T = 60\%$$

3.9. Oblique Incidence (Fresnel Equations)

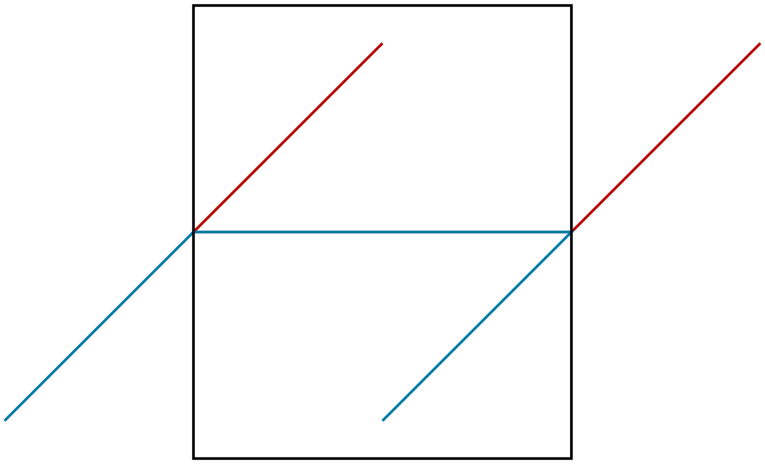
Consider a planar interface on the xy-plane, where towards $-\hat{z}$, the permittivity is ε_1, and the permeability is μ_1, and towards $+\hat{z}$, the permittivity is ε_2, and the permeability is μ_2.

---

[3] In general,

$$\begin{aligned} R &= \left(\frac{\mu_2 v_2 - \mu_1 v_1}{\mu_2 v_2 + \mu_1 v_1}\right)^2 = \left(\frac{1 - \beta}{1 + \beta}\right)^2 \\ T &= \frac{4\mu_1 v_1 \mu_2 v_2}{(\mu_1 v_1 + \mu_2 v_2)^2} = \frac{4\beta}{(1 + \beta)^2} \end{aligned}$$

It is still the case that $R + T = 1$.



Consider a propagating wave with an associated $\vec{K}_I$ and $\vec{E}_I$ (I for incidence) becoming a wave with a $\vec{K}_T$ and $\vec{E}_T$ (T for transmitted). There will also be a field $\vec{K}_R$ and $\vec{E}_R$. (R for reflected).

Define $\theta_I, \theta_T, \theta_R$ for the angles relative to the normal that the waves are at.

$$\vec{E}_I = \vec{E}_{I0} \exp\left(i\left(\vec{k}_I \cdot \vec{r} - \omega t\right)\right)$$
$$\vec{E}_R = \vec{E}_{R0} \exp\left(i\left(\vec{k}_R \cdot \vec{r} - \omega t\right)\right)$$
$$\vec{E}_T = \vec{E}_{T0} \exp\left(i\left(\vec{k}_T \cdot \vec{r} - \omega t\right)\right)$$

We will also have the magnetic fields:

$$B_I = \frac{1}{v_1}\left(\hat{k}_I \times \vec{E}_I\right)$$
$$B_R = \frac{1}{v_1}\left(\hat{k}_R \times \vec{E}_R\right)$$
$$B_T = \frac{1}{v_2}\left(\hat{k}_T \times \vec{E}_T\right)$$

We know:

$$k_I v_1 = k_R v_1 = k_T v_2 = \omega$$
$$\Rightarrow k_I = k_R = \frac{v_2}{v_1} k_T$$

$$E_- = E_I + E_R$$
$$E_+ = E_T$$
$$E_I^{\parallel} + E_R^{\parallel} = E_T^{\parallel}$$

So we will get something like:

$$(\cdots) \exp\left(i\left(\vec{k}_I \cdot \vec{r} - \omega t\right)\right) + (\cdots) \exp\left(i\left(\vec{k}_R \cdot \vec{r} - \omega t\right)\right) = (\cdots) \exp\left(i\left(\vec{k}_T \cdot \vec{r} - \omega\right)\right)$$

For the parallel components.

Therefore, on the boundary, so $r = x\hat{x} + y\hat{y}$, we need to have:

$$\vec{k}_I \cdot \vec{r} = \vec{k}_R \cdot \vec{r} = \vec{k}_T \cdot \vec{r}$$

$$x k_{Ix} + y k_{Iy}$$

$$
\begin{aligned}
&= xk_{Rx} + yk_{Ry} \\
&= xk_{Tx} + yk_{Ty}
\end{aligned}
$$

This must hold for all x and all y, so this implies that the values are equal:

$$
\begin{aligned}
k_{Ix} &= k_{Rx} = k_{Tx} \\
k_{Iy} &= k_{Ry} = k_{Ty}
\end{aligned}
$$

With this we can construct the 1st law of reflection.

The reflected, incidence and transmitted wave all lie in the same plane, the plane defined by the incident wave and normal vector of the interface. Consider if we have an incidence wave in the xz-plane, and the normal vector in the $-\hat{z}$ direction. Then, $k_{Iy} = 0$, so $k_{Ry} = 0$ and $k_{Ty} = 0$, so these are all in the xz-plane.

For the second law, due to the fact that projecting on the normal vector results in k_{Iz} and k_{Iz}, and we know they are equal because they are equal in magnitude, and the x and y components are equal by the previous boundary conditions, we have:

$$k_I \sin(\theta_I) = k_R \sin(\theta_R)$$

but since $k_I = k_R$, as they are in the same material, we have:

$$\sin(\theta_I) = \sin(\theta_R)$$

so

$$\theta_I = \theta_R$$

For the third law:

$$k_I = \frac{n_1}{n_2} k_T$$

and then, plugging into:

$$k_I \sin(\theta_I) = k_T \sin(\theta_T)$$

we have:

$$\frac{\sin(\theta_T)}{\sin(\theta_I)} = \frac{n_1}{n_2}$$

So if $n_1 < n_2$, light bends toward the normal, and if $n_1 > n_2$, light bends away from the surface normal.

So we now know the direction of the reflected and transmitted waves, but not yet the amplitude.

First, for the incident wave:

$$
\begin{aligned}
\vec{E}_I &= \vec{E}_{I0} \exp\left(i\left(\vec{k}_I \cdot \vec{r} - \omega t\right)\right) \\
\vec{B}_I &= \frac{\hat{k} \times \vec{E}_{I0}}{v_1} \exp\left(i\left(\vec{k}_I \cdot \vec{r} - \omega t\right)\right)
\end{aligned}
$$

Then, using the boundary conditions:

$$
\begin{aligned}
\varepsilon_1 E_1^{\perp} &= \varepsilon_2 \vec{E}_2^{\perp} \\
\vec{B}_1^{\perp} &= \vec{B}_2^{\perp} \\
\vec{E}_1^{\parallel} &= \vec{E}_2^{\parallel}
\end{aligned}
$$

$$\frac{1}{\mu_1}\vec{B}_1^{\parallel} = \frac{1}{\mu_2}\vec{B}_2$$

First, our variables are:

$$\vec{E}_1 = \vec{E}_I + \vec{E}_R$$
$$\vec{E}_2 = \vec{E}_T$$

Then:

$$\varepsilon_1\left(\vec{E}_{I0} + \vec{E}_{R0}\right)_\perp = \varepsilon_2\left(\vec{E}_{T0}\right)_\perp$$
$$\left(\vec{B}_{I0} + \vec{B}_{R0}\right)_\perp = \left(\vec{B}_{T0}\right)_\perp$$
$$\left(\vec{E}_{I0} + \vec{E}_{R0}\right)_\parallel = \left(\vec{E}_{T0}\right)_\parallel$$
$$\frac{1}{\mu_1}\left(\vec{B}_{I0} + \vec{B}_{R0}\right)_\parallel = \frac{1}{\mu_2}\left(\vec{B}_{T0}\right)_\parallel$$

Let us assume that the electric field is in the xz-plane. For homework, the electric field will be orthogonal to the xz-plane. With those two facts, you can create any wave through the superposition principle.

Then, we have:

$$\varepsilon_1(-E_{I0}\sin(\theta_I) + E_{R0}\sin(\theta_R)) = \varepsilon_2(-E_{T0}\sin(\theta_T))$$
$$E_{I0}\cos(\theta_I) + E_{R0}\cos(\theta_I) = E_{T0}\cos(\theta_T)$$
$$\frac{1}{\mu_1 v_1}(E_{I0} - E_{R0}) = \frac{1}{\mu_2 v_2}E_{T0}$$

The first two equations is are taking the perpendicular and parallel components to the plane of the interface, by trigonometry.

The third equation is because the magnetic field is perpendicular to the electric field and the direction of propagation, and is therefore entirely parallel to the plane of the interface, so the perpendicular boundary equation for the magnetic field immediately simplifies to $0 = 0$, and $B_\parallel = B$ for the parallel boundary condition.

To solve these equations, we will use $\theta_R = \theta_I$, and then $\sin(\theta_T) = \frac{n_1}{n_2}\sin(\theta_I)$. Notice that this means that both sides will have a $\sin(\theta_I)$, so you can remove that from both sides.

From the first equation, this will result in:

$$E_{I0} - E_{R0} = \beta E_{T0}$$

Where, like before:

$$\beta = \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\mu_1 n_2}{\mu_2 n_1}$$

Then, from the second equation:

$$E_{I0} + E_{R0} = \alpha E_{T0}$$

Where:

$$\alpha = \frac{\cos(\theta_T)}{\cos(\theta_I)}$$

This results in the Fresnel Equations:

$$E_{R0} = \frac{\alpha - \beta}{\alpha + \beta} E_{I0}$$
$$E_{T0} = \frac{2}{\alpha + \beta} E_{I0}$$

3.10. Brewster's Angle

Consider what happens when $\alpha = \beta$. Then, $\alpha - \beta = 0$, so this implies there is no reflected component (at least when the electric field is "along" the board).

Using the law of reflection:

$$\alpha = \frac{\sqrt{1 - \left(\frac{n_1}{n_2} \sin(\theta_I)\right)^2}}{\cos(\theta_I)}$$

Then, solve for $\alpha = \beta$:

$$\sin^2(\theta_B) = \frac{1 - \beta^2}{\left(\frac{n_1}{n_2}\right)^2 - \beta^2}$$

For the glass-air interface, ($n_{\text{air}} = 1$, $n_{\text{glass}} = 1.45$), this implies $\theta_B = 56°$.

For $\mu_1 = \mu_2$, we can simplify where $\beta = \frac{\mu_1 n_2}{m_2 n_1} = \frac{n_2}{n_1}$, so $\frac{n_1}{n_2} = \beta^{-1}$:

$$\begin{aligned} &\sin^2(\theta_B) \\ &= \frac{1 - \beta^2}{\left(\beta^{-1}\right)^2 - \beta^2} \\ &= \frac{1 - \beta^2}{\beta^{-2} - \beta^2} \\ &= \frac{(1 - \beta^2)\beta^2}{(\beta^{-2} - \beta^2)\beta^2} \\ &= \frac{(1 - \beta^2)\beta^2}{1 - \beta^4} \\ &= \frac{(1 - \beta^2)\beta^2}{(1 - \beta^2)(1 + \beta^2)} \\ &= \frac{\beta^2}{1 + \beta^2} \end{aligned}$$

Then, consider the fact that $\sin^2(\theta) = \frac{\tan^2(\theta)}{1 + \tan^2(\theta)}$, so:

$$\begin{aligned} \sin^2(\theta_B) &= \frac{\beta^2}{1 + \beta^2} \\ \frac{\tan^2(\theta_B)}{1 + \tan^2(\theta_B)} &= \frac{\beta^2}{1 + \beta^2} \end{aligned}$$

So, clearly $\tan^2(\theta_B) = \beta^2$, and so:

$$\begin{aligned} \tan^2(\theta_B) &= \beta^2 \\ \tan(\theta_B) &= \beta \\ \tan(\theta_B) &= \frac{n_2}{n_1} \end{aligned}$$

$$\theta_B = \arctan\left(\frac{n_2}{n_1}\right)$$

3.11. Critical angle

Consider:

$$\sin(\theta_T) = \frac{n_1}{n_2}\sin(\theta_I)$$

Consider a situation where $\theta_I = \theta_C = \sin^{-1}\left(\frac{n_2}{n_1}\right)$, (where θ_C denotes the “critical angle”).

Then:

$$\begin{aligned}\sin(\theta_T) &= \frac{n_1}{n_2}\sin\left(\arcsin\left(\frac{n_2}{n_1}\right)\right)\\ \sin(\theta_T) &= \frac{n_1}{n_2}\frac{n_2}{n_1}\\ \sin(\theta_T) &= 1\end{aligned}$$

which implies that $\theta_T = 90°$!

So consider a situation where $\theta_I > \theta_C$. This can be possible since n_1 and n_2 can occupy any positive values, so if $\frac{n_2}{n_1} < \frac{\pi}{2}$, then you can add a bit (ε) to achieve $\sin(\theta_I) = \sin\left(\arcsin\left(\frac{n_2}{n_1} + \varepsilon\right)\right)$, and this then implies that $\sin(\theta_T) > 1$.

How is this possible? If θ_T is complex! Take:

$$\sin(\theta) = \frac{\exp(i\theta) - \exp(-i\theta)}{2}$$

And we do now have an inverse. Consider the identity:

$$\cos(\theta) = \pm\sqrt{1 - \sin^2(\theta)}$$

So this implies that $\cos(\theta_T)$ is purely imaginary if $\sin(\theta_T) > 1$. So:

$$\cos(\theta_T) = \pm i\sqrt{\sin^2(\theta_T) - 1}$$

Then:

$$\vec{E}_T = E_{T0}\exp\left(i\left(\vec{k}_T \cdot \vec{r} - \omega t\right)\right)$$

Consider where $\vec{r} = x\hat{x} + z\hat{z}$, along the interface. Then:

$$\begin{aligned}&\vec{k}_t \cdot \vec{r}\\ &= k_t(x\sin(\theta_T) + z\cos(\theta_T))\\ &= k_t\left(x\sin(\theta_T) + iz\sqrt{\sin^2(\theta_T) - 1}\right)\end{aligned}$$

And therefore:

$$\vec{E}_T = \vec{E}_{T0}e^{ik_x x - \omega t}e^{-\chi z}$$

Where

$$k_x = k_T\sin(\theta_T)$$

and

$$\chi = k_t\sqrt{\sin^2(\theta_T) - 1}$$

This implies that the wave is now decaying, as we have the term $e^{-\chi z}$.

For the reflected wave:

$$E_{R0} = \frac{\alpha - \beta}{\alpha + \beta} E_{I0}$$

Where:

$$\alpha = \frac{\cos(\theta_T)}{\cos(\theta_I)}$$

and

$$\beta = \frac{\mu_1 v_1}{\mu_2 v_2}$$

Then substituting:

$$\cos(\theta_T) = i\sqrt{\sin^2(\theta_T) - 1}$$

Therefore:

$$\begin{aligned} \alpha &= i\frac{\sqrt{\sin^2(\theta_T) - 1}}{\cos(\theta_I)} \\ &= i\gamma \end{aligned}$$

Where γ is a real number. Then we have:

$$E_{R0} = \frac{i\gamma - \beta}{i\gamma + \beta} E_{I0}$$

So then:

$$\frac{I_R}{I_I} = \frac{|E_{R0}|^2}{|E_{I0}|^2} = \frac{|i\gamma - \beta|^2}{|i\gamma + \beta|^2} = 1$$

In glass, $\theta_C = \arcsin\left(\frac{1}{1.45}\right) \approx 41°$. This is used in fiber optics to efficiently transfer energy.

Conversation of energy is maintained since the energy “leaking” out through the interface has a constant amount of total energy (integrating over all space with $\frac{\varepsilon}{2} \oiiint E^2 \, \mathrm{d}v$ will be finite), so after the plane wave has been on for a while, there is no additional dissipated energy, so the reflectance being 100% still makes sense.

3.12. Electromagnetic waves in conductors

We have Maxwell’s equations:

$$\nabla \cdot \vec{E} = \frac{\rho_f}{\varepsilon}$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu \left(\vec{J} + \varepsilon \frac{\partial \vec{E}}{\partial t} \right)$$

From the continuity equation:

$$\nabla \cdot \vec{J}_f = -\frac{\partial \rho_f}{\partial t}$$

Then, with ohm's law, $\vec{J} = \sigma \vec{E}$:

$$\begin{aligned}
\nabla \cdot \vec{J}_f &= -\frac{\partial \rho_f}{\partial t} \\
-\nabla \cdot \vec{J}_f &= \frac{\partial \rho_f}{\partial t} \\
-\nabla \cdot \left(\sigma \vec{E}\right) &= \frac{\partial \rho_f}{\partial t} \\
-\sigma\left(\nabla \cdot \vec{E}\right) &= \frac{\partial \rho_f}{\partial t} \\
-\sigma\left(\frac{\rho_f}{\varepsilon}\right) &= \frac{\partial \rho_f}{\partial t}
\end{aligned}$$

This equation is separate and therefore solvable (further because it is a linear differential equation, it is an exponential):

$$\rho_f(t) = \rho_f(0) \exp\left(-\frac{\sigma}{\varepsilon} t\right)$$

With:

$$\tau = \frac{\varepsilon}{\sigma}$$

We can write this as:

$$\rho_f(t) = \rho_f(0) \exp\left(-\frac{t}{\tau}\right)$$

In a good conductor, $\tau \approx 10^{-19}\ \mathrm{s}$, but in a insulator, τ can be on the order of months.

Then, because of this very quick exponential decay, we will characterize conductors as materials such that $\tau \approx 0$, so $\rho_f = 0$.

Using Ohm's law, and the previous approximation, we can transform the equations:

$$\begin{gathered}
\nabla \cdot \vec{E} = 0 \\
\nabla \cdot \vec{B} = 0 \\
\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \\
\nabla \times \vec{B} = \mu\left(\sigma \vec{E} + \varepsilon \frac{\partial \vec{E}}{\partial t}\right)
\end{gathered}$$

Take Faraday's law:

$$\begin{aligned}
\nabla \times \vec{E} &= -\frac{\partial}{\partial t} \vec{B} \\
\nabla \times \nabla \times \vec{E} &= \nabla \times -\frac{\partial}{\partial t} \vec{B} \\
\nabla \times \nabla \times \vec{E} &= -\frac{\partial}{\partial t} \nabla \times \vec{B}
\end{aligned}$$

$$\nabla \times \nabla \times \vec{E} = -\frac{\partial}{\partial t}\left(\mu\left(\sigma\vec{E} + \varepsilon\frac{\partial \vec{E}}{\partial t}\right)\right)$$
$$\nabla \times \nabla \times \vec{E} = -\mu\left(\sigma\frac{\partial \vec{E}}{\partial t} + \varepsilon\frac{\partial^2 \vec{E}}{\partial t^2}\right)$$

Then, also alternatively:

$$\nabla \times \nabla \times \vec{E} = \nabla\left(\nabla \cdot \vec{E}\right) - \nabla^2\vec{E}$$
$$\nabla \times \nabla \times \vec{E} = \nabla(0) - \nabla^2\vec{E}$$
$$\nabla \times \nabla \times \vec{E} = -\nabla^2\vec{E}$$

So:

$$\nabla^2\vec{E} = \mu\left(\sigma\frac{\partial \vec{E}}{\partial t} + \varepsilon\frac{\partial^2 \vec{E}}{\partial t^2}\right)$$

Similarly:

$$\nabla^2\vec{B} = \mu\left(\sigma\frac{\partial \vec{B}}{\partial t} + \varepsilon\frac{\partial^2 \vec{B}}{\partial t^2}\right)$$

Let us take a plane wave propagating in the $\hat{z}$ direction:

$$\vec{E} = \hat{x}E_0\exp(i(\beta z - \omega t))$$

Also, we still have the fact that $\vec{B}$ is perpendicular to $\vec{E}$ and the direction of propagation, so:

$$\vec{B} = \hat{y}B_0\exp(i(\beta z - \omega t))$$

Therefore, we will have from the $\nabla^2\vec{E}$ equation:

$$\begin{aligned}-\beta^2E_0\exp(i(\beta z - \omega t)) &= -\mu\varepsilon\omega^2E_0\exp(i(\beta z - \omega t)) - i\mu\sigma\omega E_0\exp(i(\beta z - \omega t))\\ -\beta^2 &= -\mu\varepsilon\omega^2 - i\mu\sigma\omega\\ \beta^2 &= \mu\varepsilon\omega^2 + i\mu\sigma\omega\end{aligned}$$

Therefore:

$$\beta = \sqrt{\mu\varepsilon\omega^2 + i\mu\sigma\omega} = k_r + ik_i$$

These are:

$$k_r = \omega\sqrt{\frac{\varepsilon\mu}{2}}\sqrt{\sqrt{1 + \left(\frac{\sigma}{\varepsilon\omega}\right)^2} + 1}$$
$$k_i = \omega\sqrt{\frac{\varepsilon\mu}{2}}\sqrt{\sqrt{1 + \left(\frac{\sigma}{\varepsilon\omega}\right)^2} - 1}$$

Then, we have in sum:

$$\vec{E} = \hat{x}E_0\exp(-k_iz)\exp(i(k_rz - \omega t))$$

Based on the $\exp(-k_iz)$, the electric field is dissipating. This makes sense, since the non-infinite conductivity is converting electric energy into heat.

From this we define the skin depth:

$$d = \frac{1}{k_i}$$

Which, conceptually, is similar to a time constant τ, but for distance.

For 10 GHz waves:

Material	Skin Depth
Aluminum	0.82 µm
Copper	0.65 µm
Gold	0.79 µm
Silver	0.64 µm

To get the magnetic field, we have:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Then, we know that:

$$\begin{aligned} \vec{B} &= \hat{y} B_0 \exp(i(\beta z - \omega t)) \\ \frac{\partial \vec{B}}{\partial t} &= -\hat{y} B_0 i\omega \exp(i(\beta z - \omega t)) \\ \nabla \times \vec{E} &= \hat{y} B_0 i\omega \exp(i(\beta z - \omega t)) \end{aligned}$$

Furthermore:

$$\begin{aligned} \nabla \times \vec{E} &= \hat{y} \frac{\partial E_x}{\partial z} \\ &= \hat{y} i\beta E_0 \exp(i(\beta z - \omega t)) \end{aligned}$$

Therefore:

$$\begin{aligned} \hat{y} B_0 i\omega \exp(i(\beta z - \omega t)) &= \hat{y} i\beta E_0 \exp(i(\beta z - \omega t)) \\ B_0 \omega &= \beta E_0 \\ B_0 &= \frac{\beta}{\omega} E_0 \end{aligned}$$

Notice that β is a complex number, so this means that the magnetic field and electric field are not necessarily in phase.

Write:

$$\beta = u e^{i\varphi}$$

$$u = \omega \sqrt{\varepsilon \mu} \left(1 + \left(\frac{\sigma}{\varepsilon \omega}\right)^2\right)^{1/4}$$

And then:

$$\varphi = \arctan\left(\frac{k_i}{k_r}\right)$$

3.13. Reflection at a conducting surface

Consider an interface between two materials, in the xy-plane, where $z = 0$. Then, where $z < 0$, consider properties ε_1, μ_1, and where $z > 0$, ε_2, μ_2, and where there is a conductivity σ.

Then, we will consider a wave, where we define the following:

$$\vec{E}_I = \hat{x} E_{I0} \exp(i(k_1 z - \omega t))$$
$$\vec{E}_R = \hat{x} E_{R0} \exp(i(-k_1 z - \omega t))$$
$$\vec{E}_T = \hat{x} E_{T0} \exp(i(k_2 z - \omega t))$$

Note $k_1 \in \mathbb{R}$, but k_2 is complex!

Then:

$$\vec{B}_I = \hat{y} \frac{k_1}{\omega} E_{I0} \exp(i(k_1 z - \omega t))$$
$$\vec{B}_R = -\hat{y} \frac{k_1}{\omega} E_{R0} \exp(i(-k_1 z - \omega t))$$
$$\vec{B}_T = \hat{y} \frac{k_2}{\omega} E_{T0} \exp(i(k_2 z - \omega t))$$

Defining the variables on the left and right:

$$\vec{E}_1 = \vec{E}_I + \vec{E}_R$$
$$\vec{E}_2 = \vec{E}_T$$

Considering boundary directions:

$$E_{I0} + E_{R0} = E_{T0}$$
$$\frac{k_1}{\mu_1 \omega}(E_{I0} - E_{R0}) = \frac{k_2}{\mu_2 \omega} E_{T0}$$

Define β:

$$\beta = \frac{\mu_1 k_2}{\mu_2 k_1}$$

Then, the solution is:

$$E_{R0} = \frac{1 - \beta}{1 + \beta} E_{I0}$$
$$E_{T0} = \frac{2}{1 + \beta} E_{I0}$$

Now, split k_2:

$$k_2 = k_r + i k_i$$

$$k_r = \omega \sqrt{\frac{\varepsilon\mu}{2}} \sqrt{\sqrt{1 + \left(\frac{\sigma}{\varepsilon\omega}\right)^2} + 1}$$

$$k_i = \omega \sqrt{\frac{\varepsilon\mu}{2}} \sqrt{\sqrt{1 + \left(\frac{\sigma}{\varepsilon\omega}\right)^2} - 1}$$

Then, if σ, the conductivity, is very large, this implies that β is very large, and then, as $\lim_{\beta \to \infty} \frac{1-\beta}{1+\beta} = -1$, nearly all of the electric field is reflected. This is why metal is reflective. For instance, silver is 95% reflective.

3.14. Frequency Dependent susceptibility

In a linear isotropic homogenous material:

$$\vec{P} = \varepsilon_0 \chi_e \vec{E}$$

χ_e is the susceptibility.

But, χ_e can often depend on frequency, so $\chi_e = \chi_e(\omega)$. Previously, in statics, we were only concerned about statics, but we can broaden the scope.

But, this χ_e was based on the Lorentz model where the electrons are considered to be connected by springs, but those springs don't instantly respond, nor do they instantly settle! Furthermore, there will also be a resonance frequency for the springs.

Previously the dipole moment was defined by:

$$\vec{p} = \alpha \vec{E}$$

where α is the atomic polarizability.

Then:

$$\vec{P} = N\vec{p} = N\alpha\vec{E} = \varepsilon_0 \chi_e \vec{E}$$

Where N is the atomic density.

So, therefore:

$$\chi_e = \frac{N\alpha}{\varepsilon_0}$$

For a general problem we have:

$$F = F_r + F_{\text{ext}} + F_d$$

Where F_r is the restoring force, F_{ext} is the external force from the electric field, and F_d is the frictional force, which usually comes from either radiation or interactions with other atoms in the lattice.

$$\begin{aligned} F_r &= -m\omega_0^2 x \\ \omega_0 &= \sqrt{\frac{k}{m}} \\ F_{\text{ext}} &= qE \\ F_d &= -m\gamma \frac{\mathrm{d}x}{\mathrm{d}t} \end{aligned}$$

So we have 2 material parameters, ω_0, the resonance frequency, and γ.

Note that for a spring, $\omega_0 = \sqrt{k/m}$.

Therefore, we have:

$$F - F_d \qquad\qquad - F_r = F_{\text{ext}}$$

$$m\frac{\mathrm{d}^2 x}{\mathrm{d}t} + m\gamma\frac{\mathrm{d}x}{\mathrm{d}t} + m\omega_0^2 x = F_{\text{ext}} = qE(t)$$

$$\frac{\mathrm{d}^2x}{\mathrm{d}t}+\gamma\frac{\mathrm{d}x}{\mathrm{d}t}\qquad+\omega_0^2x=\frac{qE(t)}{m}$$

So, this is clearly a non-homogenous linear differential equation, so finding the general solution is easy after finding any single solution for some particular F_{ext}. It is also clearly a damped harmonic oscillator driven by the electric field.

We will write:

$$\vec{E}(z,t)=\hat{x}\ \mathrm{Re}\left\{\tilde{E}\exp(i(kz-\omega t))\right\}$$

This does not actually restrict the electric field, since we can take the Fourier transform to decompose the field into its frequency components.[4] Also, note that you can rotate the problem so that $\hat{x}$ is the direction of the electric field, and $\hat{z}$ is the direction of propagation. This also implies that $\hat{x}$ is also the direction of displacement.

Now, since all the forces are along the $\hat{x}$ direction, we can just treat everything as a scalar. Furthermore, since this is linear differential equation, we know that the input and output frequencies will be the same. So then:

$$\begin{aligned}E(z,t)&=\mathrm{Re}\left\{\tilde{E}\exp(i(kz-\omega t))\right\}\\ \tilde{E}&=E_0\exp(i\theta)\\ x(z,t)&=\mathrm{Re}\{\tilde{x}\exp(i(kz-\omega t))\}\end{aligned}$$

By linearity, we will consider:

$$\begin{aligned}E(z,t)&=\tilde{E}\exp(i(kz-\omega t))\\ x(z,t)&=\tilde{x}\exp(i(kz-\omega t))\end{aligned}$$

Taking derivatives:

$$\begin{aligned}x'(z,t)&=-i\omega\tilde{x}\exp(i(kz-\omega t))\\ x''(z,t)&=-\omega^2\tilde{x}\exp(i(kz-\omega t))\end{aligned}$$

Now we can plug this into the differential equation:

$$\begin{aligned}\frac{\mathrm{d}^2x}{\mathrm{d}t}+\gamma\frac{\mathrm{d}x}{\mathrm{d}t}\qquad\qquad\qquad\qquad\qquad\qquad+\omega_0^2x&=\frac{qE(t)}{m}\\ -\omega^2\tilde{x}\exp(i(kz-\omega t))+\gamma(-i\omega\tilde{x}\exp(i(kz-\omega t)))+\omega_0^2\tilde{x}\exp(i(kz-\omega t))&=\frac{q\tilde{E}\exp(i(kz-\omega t))}{m}\\ -\omega^2\tilde{x}+\gamma(-i\omega\tilde{x})\qquad\qquad\qquad\qquad\qquad\qquad+\omega_0^2\tilde{x}&=\frac{q\tilde{E}}{m}\end{aligned}$$

Then, to solve for $\tilde{x}$:

$$\begin{aligned}-\omega^2\tilde{x}+\gamma(-i\omega\tilde{x})+\omega_0^2\tilde{x}&=\frac{q\tilde{E}}{m}\\ (\omega_0^2-\omega^2-i\gamma\omega)\tilde{x}&=\frac{q\tilde{E}}{m}\\ \tilde{x}&=\frac{q/m}{\omega_0^2-\omega^2-i\gamma\omega}\tilde{E}\end{aligned}$$

From this, as $\tilde{p}=q\tilde{x}$, and $\tilde{p}=\alpha\tilde{E}$, then:

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[4] And with complex k, the field even need not continue forever.

$$\alpha = \frac{q^2/m}{\omega_0^2 - \omega^2 - i\gamma\omega}$$

From above:

$$\chi_e = \frac{N\alpha}{\varepsilon_0}$$

Therefore, in sum:

$$\chi_e(\omega) = \frac{Nq^2}{\varepsilon_0 m(\omega_0^2 - \omega^2 - i\gamma\omega)}$$

$$\varepsilon_r(\omega) = 1 + \chi_e(\omega)$$

$$\varepsilon(\omega) = \varepsilon_0\varepsilon_r(\omega)$$

This works best in gasses, in solids the band structure messes things up.

If $\omega \approx \omega_0$, then you have a very high χ_e. Then, the imaginary part of χ_e represents the damping of the electric field through the material.

Note that then, $k = (n\omega)/c = \omega\sqrt{1+\chi_e}/c = k_r + ik_i$

Because if χ_e is complex, k is as well.

Then, the intensity is $I \propto |E|^2 \propto e^{-2k_i z}$, and then there is a different α, the absorption coefficient, defined as $\alpha = 2k_i$.

3.15. Free Electrons

This can be considered the same as above but with a spring constant of 0, and therefore $\omega_0 = 0$. Further, $q = e$. So:

$$\varepsilon_r(\omega) = 1 - \frac{Ne^2}{\varepsilon_0 m(\omega^2 + i\gamma\omega)}$$

If we assume $\gamma \ll \omega$:

$$\approx 1 - \frac{Ne^2}{m\varepsilon_0\omega^2}$$

Then, ε_r is negative when $\omega < \omega_p$ (p is for plasma, where plasma is referring to the electron plasma formed from free electrons):

$$\omega_p = \sqrt{\frac{Ne^2}{m\varepsilon_0}}$$

Then, this means that $n = \sqrt{\varepsilon_r}$ is purely imaginary, so the waves formed are evanescent waves.

$$\lambda_p = \frac{2\pi c}{\omega_p} \approx 120 \text{ nm}$$